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Boolean Valued Models, Boolean Valuations, and Löwenheim-Skolem Theorems

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Abstract
Boolean-valued models for first-order languages generalize two-valued models, in that the value range is allowed to be any complete Boolean algebra instead of just the Boolean algebra 2. Boolean-valued models are interesting in multiple aspects: philosophical, logical, and mathematical. The primary goal of this paper is to extend a number of critical model-theoretic notions and to generalize a number of important model-theoretic results based on these notions to Boolean-valued models. For instance, we will investigate (first-order) Boolean valuations, which are natural generalizations of (first-order) theories, and prove that Boolean-valued models are sound and complete with respect to Boolean valuations. With the help of Boolean valuations, we will also discuss the Löwenheim-Skolem theorems on Boolean-valued models.

Keywords Boolean-valued models · Non-classical model theory · Löwenheim-Skolem theorems

1 Introduction

Traditionally, a model of a first order language $\mathscr{L}$ has as its value range the complete Boolean algebra $2 = \{0, 1\}$. Logical symbols in the language are interpreted as operations on the Boolean algebra: conjunction as binary meet, disjunction as binary join, negation as Boolean complement, universal quantifier as infinite meet and existential quantifier as infinite join. A natural way to generalize the traditional models, then, is to instead of just using the complete Boolean algebra 2 as the value range, use arbitrary complete Boolean algebra as value ranges.

Boolean-valued models are worth studying for a variety of reasons. From a philosophical perspective, Boolean-valued models have interesting applications to the

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phenomenon of vagueness. The supervaluation models, which are used in the standard approach to vagueness, can be shown to be a special type of Boolean-valued models (Theorem 3.1). In fact, we can show that there is a duality between the class of supervaluation models and a subclass of true identity Boolean-valued models (Theorem 3.3). Also, two important features of Boolean-valued models - that they are degree-theoretic and that they induce classical logic - let them give rise to attractive theories of different types of vagueness[1]. Moreover, since the logic of Boolean-valued models is both classical and non-bivalent, they are particularly useful in illustrating certain points in the philosophy of model theory. For example, it seems to serve as a strong case against the claim that our classical rules of inferences pin down uniquely the range of semantic values ([3]).

From a logical perspective, a number of important model-theoretic results on two-valued models can be shown to be special cases of more generalized theorem on Boolean-valued models. A (relatively) well-known example is that the Łos' Theorem on ultraproducts is a specific instance of a more general theorem on Boolean-valued models that satisfy some special condition[2]. In this paper, we will also show that the Löwenheim-Skolem theorems are specific cases of some more general theorems on Boolean-valued models. Boolean-valued models are also useful for model construction purposes. For example, the ultraproduct construction is a special case of the combination of the direct product construction and the quotient construction on Boolean-valued models (see [19] or [20]). Another example is Boolean ultrapowers, which generalize the regular ultrapower construction to any complete Boolean algebra, rather than only on power set algebra (see [12] or [8]).

Finally, from a mathematical perspective, Boolean-valued models are famous for their usefulness in the context of set theory. Introduced by Dana Scott, Robert Solovay and others, Boolean-valued models for the language of set theory are used to give semantics to Paul Cohen's syntactic forcing, which is a method for obtaining independence results (see [2] or Jech [10]). Recent works have shown that Boolean-valued models, via their connection with forcing, can also be used to yield fruitful results on operator algebras[3].

Despite their utility, Boolean-valued models, as a subject on their own, have not been as well-studied as the two-valued models. On two-valued models there exists a full-fledged, robust and fruitful theory - the entirety of model theory, roughly speaking, that is based on important basic notions like "diagram", "submodel", "elementary", etc. Few of these notions, to the author's knowledge, have been generalized to Boolean-valued models, and so are the case with the many model-theoretic results based on these notions. There are a number of natural questions on the model-theoretic properties of Boolean-valued models that awaits answers: What is the diagram/elementary diagram

[1] For an application to the general phenomenon of vagueness, see Mcgee and Mclaughlin [13]. For an application to mereological indeterminacy, see [22]. For an application to indeterminacy in identity, see [20].

[2] In particular, the condition of being "witnessing", as defined in Definition 3.4. For a proof of the generalized Łos' Theorem, see Hamkins [7] or Viale [19]. For a proof of a more general version of this theorem, see Wu [21]. For a form of Łos' Theorem on Heyting-valued models, see Aratake [1].

[3] Jech [11] and Takeuti [17] have shown that there's a duality translating the commutative $C*$ algebras to the family of B-names for complex numbers in V^B. Viale [18] extends this duality to arbitrary Polish spaces.

of a Boolean-valued model? What does it mean for a Boolean-valued model to be a submodel/elementary submodel of another? Do Löwenheim-Skolem Theorems hold on all Boolean-valued models? etc. The primary goal of this paper is to answer these questions.

When we only have two truth values, the diagram of a model is a set of sentences, and therefore a theory. But when there are more than two truth values, the “diagram” of a model, if we want it to be something close to what we have in the two-valued case, cannot be just a theory. The natural suggestion is that the diagram is a set of ordered pairs whose first component is a sentence and second component is a truth value. In this paper, we will call a set of this form a “Boolean valuation". (First-order) Boolean valuations are natural generalizations of (first-order) theories. The first major result of this paper (Theorem 4.7.1) is that (under our definition of consistency), Boolean-valued models are sound and complete with respect to Boolean-valuations, which is a theorem that generalizes the known result that Boolean-valued models are sound and complete with respect to first-order theories (see, for example, [15]). Corollaries to this theorem include the compactness theorem (Corollary 4.7.2) on Boolean valuations and the (weaker version) of Downward-Löwenheim-Skolem theorem on Boolean valuations (Corollary 4.7.3).

With the notion of “Boolean valuation", we are then able to define notions like “diagram"(Definition 5.6), “elementary diagram"(Definition 5.8), etc., and prove the equivalence theorems between diagrams and submodels (Theorem 5.4), elementary diagrams and elementary submodels (Theorem 5.6), etc. The next major result is the generalization of (the stronger version) Downward-Löwenheim-Skolem theorem to witnessing Boolean-valued models (Theorem 5.7), and that it does not necessarily hold on non-witnessing Boolean-valued models (Theorem 5.8).

For the discussion of the Upward-Löwenheim-Skolem theorems to be non-trivial, we will have to look at a special type of Boolean-valued models, the ones that define identity in the standard, or true way (Definition 3.2). We will investigate which kind of Boolean valuations corresponds to the “true identity" models. The third major result (Theorem 6.7) is that true identity Boolean-valued models are sound and complete with respect to Boolean valuations that “respect identity" (Definition 6.4). From there, we will show the Upward-Löwenheim-Skolem theorems on true identity Boolean-valued models (Theorems 7.6, 7.7).

We organize this paper as follows: in Section 2, we introduce Boolean-valued models. In Section 3 we discuss the connection between supervaluation models and Boolean-valued models. In particular, we prove that supervaluation models are equivalent to a special type of Boolean-valued models. In Section 4, we first review the proof of the theorem that Boolean-valued models are sound and complete with respect to first-order theorems, and then in 4.2, we introduce Boolean valuations, define their consistency condition, and prove that Boolean-valued models are sound and complete with respect to first-order Boolean valuations. In Section 5, with the help of Boolean valuations, we extend basic model theoretic notions like “diagram", “submodel", “elementary embedding" to Boolean-valued models, prove the equivalence theorems, and prove the (stronger version of) Downward-Löwenheim-Skolem theorem on witnessing Boolean-valued models. We will also study chains of models and generalize the Elementary Chain Theorem to the Boolean-valued case. In Section 6, we will investigate the true identity Boolean-value models and prove their soundness and completeness theorems. Finally, in Section 7, we discuss the Upward-Löwenheim-Skolem theorems on Boolean-valued models.

2 Boolean Valued Models

We assume here that the reader already has some basic knowledge about Boolean algebras and model theory. For a detailed introduction of Boolean algebras, see Givant and Halmos [6].

In this paper, we will use the symbol "$\sqcap$" for lattice meet(infimum), "$\sqcup$" for lattice join(supremum), and "$-$" for Boolean complement. A Boolean algebra B is κ-complete (where κ is a cardinal) just in case for any subset $D \subseteq B$ such that $|D| < \kappa$, both the supremum of D, $\bigsqcup D$, and the infimum of D, $\bigsqcap D$, exist in B. A Boolean algebra B is complete just in case for any κ, B is κ-complete.

Definition 2.1 Let $\mathscr{L}$ be an arbitrary first order language. For simplicity, we assume that $\mathscr{L}$ has no function symbols, but only relation symbols and constants.[4] Let B be a complete Boolean algebra. A B-valued[5] model $\mathfrak{A}$ for the language $\mathscr{L}$ consists of[6] :

1. A universe A of elements;
2. The B-value of the identity symbol: a function $[\![=]\!]^{\mathfrak{A}} : A^2 \to B$;
3. The B-values of the relation symbols: (let P be a n-ary relation) $[\![P]\!]^{\mathfrak{A}} : A^n \to B$;
4. The B-values of the constant symbols: (let c be a constant) $[\![c]\!]^{\mathfrak{A}} \in A$.

And it needs to satisfy:

1. For the B-value of the identity symbol[7]: for any $a_1, a_2, a_3 \in A$

$$[\![a_1 = a_1]\!]^{\mathfrak{A}} = 1_B \tag{1}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} = [\![a_2 = a_1]\!]^{\mathfrak{A}} \tag{2}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} \sqcap [\![a_2 = a_3]\!]^{\mathfrak{A}} \leqslant [\![a_1 = a_3]\!]^{\mathfrak{A}} \tag{3}$$

2. For the B-value of relation symbols: let P be an n-ary relation; for any $\langle a_1, ..., a_n\rangle, \langle b_1, ..., b_n\rangle \in A^n$,

$$[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}} \sqcap \left(\bigsqcap_{1\leqslant i\leqslant n} [\![a_i = b_i]\!]^{\mathfrak{A}}\right) \leqslant [\![P(b_1, ..., b_n)]\!]^{\mathfrak{A}} \tag{4}$$

[4] Our theory can be easily generalized to first order languages with function symbols, as functions can always be treated as relations that satisfy special conditions.

[5] Here and in the following, when $\mathfrak{A}$ is a B-valued model, we will call B the *value range* of $\mathfrak{A}$. It is important to note that B is the *codomain*, not the *range*, of the B-values of the relation/identity symbols. In particular, it is possible that for some value p in B, there is no formula whose value in the model is p.

[6] Our definition of Boolean-valued models is the standard one. You can find the same definition in many other places, including, Bell [2], Button and Walsh [3], Hamkins and Seabold [8], etc.

[7] Here and in the following, when the context is clear, we use $[\![a_i = a_j]\!]^{\mathfrak{A}}$ to abbreviate $[\![=]\!]^{\mathfrak{A}}(a_i, a_j)$, and similarly for cases of the relation symbols.

Given a B-valued model $\mathfrak{A}$ for $\mathscr{L}$, we define satisfaction in $\mathfrak{A}$ as follows:

Definition 2.2 Let Var be the set of all variables. (We will use $v_1, v_2, \ldots$ to range over variables.) An assignment on $\mathfrak{A}$ is a function from Var to A. Given a assignment x on $\mathfrak{A}$, we define the value of an open formula of $\mathscr{L}$ in $\mathfrak{A}$ under assignment x as follows.

1. We first define the value of terms in $\mathfrak{A}$:
   (a) Let v_i be a variable. Then $[\![v_i]\!]^{\mathfrak{A}}[x] = x(v_i) = x_i$[8].
   (b) Let c be a constant. Then $[\![c]\!]^{\mathfrak{A}}[x] = [\![c]\!]^{\mathfrak{A}}$.
2. We then define the value of atomic formulas in $\mathfrak{A}$:
   (a) Let t_1, t_2 be terms (a term is either a variable or a constant). Then $[\![t_1 = t_2]\!]^{\mathfrak{A}}[x] = [\![a_i = a_j]\!]^{\mathfrak{A}}$, where $a_i = [\![t_1]\!]^{\mathfrak{A}}[x]$ and $a_j = [\![t_2]\!]^{\mathfrak{A}}[x]$.
   (b) Let $t_1, \ldots, t_n$ be terms. Then $[\![P(t_1, \ldots, t_n)]\!]^{\mathfrak{A}}[x] = [\![P(a_i, \ldots, a_k)]\!]^{\mathfrak{A}}$, where $a_i = [\![t_1]\!]^{\mathfrak{A}}[x], \ldots, a_k = [\![t_n]\!]^{\mathfrak{A}}[x]$.
3. We finally define the value of complex formulas in $\mathfrak{A}$:
   (a) Let ϕ be a formula. Then $[\![\neg\phi]\!]^{\mathfrak{A}}[x] = -[\![\phi]\!]^{\mathfrak{A}}[x]$.
   (b) Let ϕ, ψ be formulas. Then $[\![\phi \wedge \psi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{\mathfrak{A}}[x] \sqcap [\![\psi]\!]^{\mathfrak{A}}[x]$.
   (c) Let ϕ, ψ be formulas. Then $[\![\phi \vee \psi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{\mathfrak{A}}[x] \sqcup [\![\psi]\!]^{\mathfrak{A}}[x]$.
   (d) Let ϕ be a formula. Then $[\![\exists v_i \phi]\!]^{\mathfrak{A}}[x] = \bigsqcup_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)]$, where $x(v_i/a)$ is the assignment on $\mathfrak{A}$ that takes v_i to a and agrees with x everywhere else.
   (e) Let ϕ be a formula. Then $[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x] = \bigsqcap_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)]$, where $x(v_i/a)$ is the assignment on $\mathfrak{A}$ that takes v_i to a and agrees with x everywhere else.

Clearly, both $[\![\exists v_i \phi]\!]^{\mathfrak{A}}[x]$ and $[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x]$ are well-defined as B is assumed to be complete.
It is easy to see that traditional two-valued models for first order languages are just special cases of Boolean valued models, when we require B to be the two-element Boolean algebra 2 and that the interpretation of the identity symbol is the true identity function on the universe[9].
In the following, like in the case of atomic formulas, when the context is clear, we will occasionally use $[\![\phi(x_1, \ldots, x_n)]\!]^{\mathfrak{A}}$, instead of $[\![\phi(v_1, \ldots, v_n)]\!]^{\mathfrak{A}}[x]$.

Theorem 2.1 Let $\mathfrak{A}$ be a B-valued model for $\mathscr{L}$. For any formula $\phi(v_1, \ldots, v_n)$ in $\mathscr{L}$, any assignments x, y on $\mathfrak{A}$,

$$[\![\phi(x_1, \ldots, x_n)]\!]^{\mathfrak{A}} \sqcap \left(\bigsqcap_{1 \leqslant i \leqslant n} [\![x_i = y_i]\!]^{\mathfrak{A}} \right) \leqslant [\![\phi(y_1, \ldots, y_n)]\!]^{\mathfrak{A}}$$

Proof By a straightforward induction on the complexity of $\phi(v_1, \ldots, v_n)$. □

[8] Here and in the following, given an assignment x, we will use x_i to abbreviate $x(v_i)$.

[9] We assume that the reader has some basic knowledge of traditional two-valued models. For a detailed introduction on model theory, see Chang and Keisler [4], or Hodges [9].

3 Supervaluationism

In this section, we show that supervaluation models are special cases of Boolean-valued models. In particular, we show that every supervaluation model is equivalent to an elementary submodel of the direct product of the precisifications. Also, the class of supervaluation models is equivalent to a subclass of true identity Boolean-valued models: roughly, any supervaluation model has a canonical Boolean counterpart whose value range is the powerset algebra of the set of all precisifications.

Definition 3.1 A *supervaluation* model $\mathfrak{S}$ for $\mathscr{L}$ is a pair $\langle A, \Sigma \rangle$ such that A is a domain of elements and $\Sigma = \{\sigma_i \mid i \in I\}$ is a collection of two-valued interpretation functions (indexed by I). In particular[10],

1. Let c be a constant in $\mathscr{L}$. For some $a \in A$, for any $i \in I$, $\sigma_i(c) = a$.
2. Let P be a n-ary relation in $\mathscr{L}$. For any $i \in I$, $\sigma_i(P) = R_i \subseteq A^n$.

For each $i \in I$, $\mathfrak{A}_i$ is the two-valued model for $\mathscr{L}$ with domain A and interpretation function σ_i. Every $\mathfrak{A}_i$ is called a *precisification* in $\mathfrak{S}$.
For any formula $\phi(v_1, ..., v_n)$ in $\mathscr{L}$, and any assignment function $x : Var \to A$,

$$[\![\phi]\!]^{\mathfrak{S}}[x] = \begin{cases} \textit{(super)true} & \text{if for every } \; i \in I, \mathfrak{A}_i \models \phi[x]; \\ \textit{(super)false} & \text{if for every } \; i \in I, \mathfrak{A}_i \models \neg\phi[x]; \\ \textit{undefined} & \text{if otherwise} \end{cases}$$

Definition 3.2 A B-valued model $\mathfrak{A}$ is a *true identity model* just in case $[\![=]\!]^{\mathfrak{A}} : A \times A \to B$ is the real identity function on $A \times A$, i.e. for any $a, b \in A$, if a and b are not the same element, then $[\![a = b]\!]^{\mathfrak{A}} = 0_B$.

Definition 3.3 Given a supervaluation model $\mathfrak{S} = \langle A, \{\sigma_i \mid i \in I\}\rangle$, we construct a $P(I)$-valued model $\mathfrak{M}^{\mathfrak{S}}$ for $\mathscr{L}$ as follows (where $P(I)$ is the powerset of I endowed with the powerset algebra):

1. The domain of $\mathfrak{M}^{\mathfrak{S}}$ is A.
2. $[\![=]\!]^{\mathfrak{M}^{\mathfrak{S}}} : A^2 \to P(I)$ is such that for any $a, b \in A$, $[\![a = b]\!] = \emptyset$ if a and b are not the same element, and $[\![a = b]\!] = I$ if a and b are the same element.
3. Let c be a constant in $\mathscr{L}$, $[\![c]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \sigma_i(c)$, for any $i \in I$.
4. Let P be a n-ary relation in $\mathscr{L}$. $[\![P]\!]^{\mathfrak{M}^{\mathfrak{S}}} : A^n \to P(I)$ is such that for any $a_1, ..., a_n \in A$, $[\![P(a_1, ..., a_n)]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{A}_i \models P(a_1, ..., a_n)\}$.

[10] We assume here that a constant is always interpreted as the same individual in all precisifications. Although this is the default assumption in most standard formulations of supervaluationism (as in, for example, [5] or [16]), we are aware of the need for loosing this assumption in certain situations. The results we present below can be generated to more general definitions of supervaluation models, including ones in which constants can have different referents in different precisifications, and even ones in which the domains of different precisifications can be different. Due to the lack of space we will not present the details here. Roughly, in cases where we have constants without a unvarying referent, we can simply regard a constant as a unary predicate that satisfies the special condition that its extension is a singleton. And in cases where we have precisifications with different domains, we can simply pretend that all precisifications have the union of all the domains as their domain, and have an existential predicate whose extension in each precisification is the actual domain of the precisification, and have the quantifiers be restricted to what satisfies the existential predicate in each precisification.

It is easy to check that $\mathfrak{M}^{\mathfrak{S}}$ is a true identity Boolean-valued model.

Theorem 3.1 For any formula $\phi(v_1, ..., v_n)$ in $\mathscr{L}$, and any assignment function $x : Var \to A$,

$$[\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}}[x] = \{i \in I \mid \mathfrak{A}_i \models \phi[x]\}$$

Proof By induction on the complexity of ϕ. The atomic cases are covered by the definition of $\mathfrak{M}^{\mathfrak{S}}$. The cases for sentential connectives are straightforward. For existential quantifier,

$$\begin{aligned}
[\![\exists v_j \phi]\!]^{\mathfrak{M}^{\mathfrak{S}}}[x] &= \bigcup_{a \in A} [\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}}[x(v_j/a)] \\
&= \bigcup_{a \in A} \{i \in I \mid \mathfrak{A}_i \models \phi[x(v_j/a)]\} \\
&= \{i \in I \mid \mathfrak{A}_i \models \exists v_j \phi[x]\}
\end{aligned}$$

The case for universal quantifier is similar. □

As a result, the supervaluation model $\mathfrak{S}$ is essentially equivalent to its Boolean counterpart $\mathfrak{M}^{\mathfrak{S}}$. They have the same domain, and for any ϕ in $\mathscr{L}$, the degree to which ϕ is true in $\mathfrak{M}^{\mathfrak{S}}$ is the set of all precisifications in $\mathfrak{S}$ in which ϕ is true. Therefore, ϕ is (super)true in $\mathfrak{S}$ iff $[\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}} = I$, which is the top value in $P(I)$, and ϕ is (super)false in $\mathfrak{S}$ iff $[\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \emptyset$, which is the bottom value in $P(I)$. Since all classical tautologies have value 1 in every Boolean-valued model, all classical tautologies are (super)-true in every supervaluation model.
We next show that $\mathfrak{S}$ is an elementary submodel of the direct product of all the precisifications.

Theorem 3.2 Let $\mathfrak{S} = \langle A, \{\sigma_i \mid i \in I\}\rangle$ be a supervaluation model. Let $\{\mathfrak{A}_i \mid i \in I\}$ be its set of precisifications. Let $\prod_{i \in I} \mathfrak{A}_i$ be their direct product (Definition 5.10). $\mathfrak{M}^{\mathfrak{S}}$ is an elementary submodel (Definition 5.7) of $\prod_{i \in I} \mathfrak{A}_i$.

Proof Clearly $P(I)$ and $\prod_{i \in I} 2$ are isomorphic. The elementary embedding is the function $f : A \to \prod_{i \in I} A_i$ that takes any $a \in A$ to $\langle a\rangle_{i \in I}$.
We just need to show that for any formula $\phi(v_1, ..., v_n)$ in $\mathscr{L}$, any $a_1, ..., a_n \in A$,

$$[\![\phi(a_1, ..., a_n)]\!]^{\mathfrak{M}^{\mathfrak{S}}} = [\![\phi(\langle a_1\rangle_{i \in I}, ..., \langle a_n\rangle_{i \in I})]\!]^{\prod_{i \in I} \mathfrak{A}_i}$$

By the Direct Product Theorem (Theorem 5.10), $[\![\phi(\langle a_1\rangle_{i \in I}, ..., \langle a_n\rangle_{i \in I})]\!]^{\prod_{i \in I} \mathfrak{A}_i} = \{i \in I \mid \mathfrak{A}_i \models \phi(a_1, ..., a_n)\} = [\![\phi(a_1, ..., a_n)]\!]^{\mathfrak{M}^{\mathfrak{S}}}$, by Theorem 3.1. □

Definition 3.4 Let $\mathfrak{A}$ be a B-valued model for the language $\mathscr{L}$. Then $\mathfrak{A}$ is *witnessing*[11] just in case for any formula $\phi(u, v_1, ..., v_n)$ of $\mathscr{L}$, any $a_1, ..., a_n \in A$, there is an $a \in A$ such that

$$[\![\exists u \phi(u, v_1, ..., v_n)]\!]^{\mathfrak{A}}[a_1, ..., a_n] = [\![\phi(u, v_1, ..., v_n)]\!]^{\mathfrak{A}}[a, a_1, ..., a_n]$$

Observation 3.2.1 Let $\mathfrak{S} = \langle A, \{\sigma_i \mid i \in I\}\rangle$ be a supervaluation model. $\mathfrak{M}^{\mathfrak{S}}$ may not be a witnessing model, although $\prod_{i \in I} \mathfrak{A}_i$ is always witnessing. The latter is because direct products always inherit the property of being witnessing, which follows from Theorem 5.10. It is easy to construct examples of the former. For example, we can let a unary predicate P be such that it has a non-empty extension in every $\mathfrak{A}_i$ in $\mathfrak{S}$, yet there is no $a \in A$ that is in the extension of P in in every $\mathfrak{A}_i$ in $\mathfrak{S}$. Then $\exists v_i P(v_i)$ will have value I in $\mathfrak{M}^{\mathfrak{S}}$ without a witness.

Corollary 3.2.1 (to Theorem 4.1) Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. $T \vdash \phi$ if and only if for any supervaluation model $\mathfrak{S}$, if every member of T is (super)true in $\mathfrak{S}$, then ϕ is (super)true in $\mathfrak{S}$.

We have shown that every supervaluation model is equivalent to a true identity Boolean-valued model. Our next goal is to establish a duality between the class of supervaluation models and a subclass of true identity models.

Definition 3.5 Let B and C be two complete Boolean algebras and let $\mathfrak{A}$ be a B-valued model. $\mathfrak{A}$ is *C-embeddable* just in case there is an embedding(monomorphism) $f : B \to C$ such that for any formula $\phi(v, v_1, ..., v_n)$, $a_1, ..., a_n \in A$

$$f([\![\exists v \phi]\!]^{\mathfrak{A}})[a_1, ..., a_n] = \bigsqcup_{a \in A} f([\![\phi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n])$$
$$f([\![\forall v \phi]\!]^{\mathfrak{A}})[a_1, ..., a_n] = \bigsqcap_{a \in A} f([\![\phi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n])$$

Theorem 3.3 Let $\mathfrak{A}$ be a B-valued model. Then $\mathfrak{A}$ is equivalent to a supervaluation model just in case $\mathfrak{A}$ is a true identity model and is $\mathscr{P}(I)$-embeddable, for some powerset algebra $\mathscr{P}(I)$.

Proof Let $\mathfrak{S} = \langle A, \{\sigma_i \mid i \in I\}\rangle$ be a supervaluation model and let $\mathfrak{M}^{\mathfrak{S}}$ be the $\mathscr{P}(I)$-valued model as defined in Definition 3.3. Then $\mathfrak{M}^{\mathfrak{S}}$ is a true identity model and is $\mathscr{P}(I)$-embeddable by the identity function.

For the other direction, let $\mathfrak{A}$ be a true identity B-valued model that is $\mathscr{P}(I)$-embeddable, for some powerset algebra $\mathscr{P}(I)$, by an embedding $f : B \to \mathscr{P}(I)$. For each $i \in I$, we construct a 2-valued model $\mathfrak{A}_i$ with domain A as follows:

[11] Witnessing Boolean-valued models are important because they are the ones on which the Łos' Theorem (Theorem 5.2) holds, while Łos' Theorem does not hold on Boolean-valued models in general (See [21] or [19]). For a topological characterization of the property of being witnessing, see [14]. Some people, including Hamkins and Seabold [8], Jech [10] and Viale [14], call witnessing models "full" models instead. We use the term "witnessing" here because the term "full" is sometimes used to refer to models that satisfy a different condition (Definition 6.3). A hidden misunderstanding on this subject seems to be that the two definitions coincide. But in fact they are not. We will show in Section 6 that full models, defined in terms of antichains, are all witnessing models, yet the converse does not hold.

1. Let c be a constant in $\mathscr{L}$, $[\![c]\!]^{\mathfrak{A}_i} = [\![c]\!]^{\mathfrak{A}} \in A$.
2. Let P be a n-ary relation in $\mathscr{L}$. For any $a_1, ..., a_n \in A$, $\mathfrak{A}_i \models P(a_1, ..., a_n)$ iff $i \in f([\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}})$.

Let $\mathfrak{S}$ be the supervaluation model with precisifications $\{\mathfrak{A}_i \mid i \in I\}$. Let $\mathfrak{M}^{\mathfrak{S}}$ be the $\mathfrak{S}$-induced $\mathscr{P}(I)$-valued model as defined in Definition 3.3. Then for any formula $\phi(v_1, ..., v_n)$ in $\mathscr{L}$, any $a_1, ..., a_n \in A$,

$$[\![\phi(a_1, ..., a_n)]\!]^{\mathfrak{M}^{\mathfrak{S}}} = f([\![\phi(a_1, ..., a_n)]\!]^{\mathfrak{A}})$$

The claim can be proven by induction on the complexity of ϕ. The atomic cases are governed by the definition of $\mathfrak{S}$. The cases for connectives hold because f is an embedding, and the cases for quantifiers hold because f witnesses that $\mathfrak{A}$ is $\mathscr{P}(I)$-embeddable.
As a result, every value in $\mathscr{P}(I)$ that is possibly “used" in $\mathfrak{M}^{\mathfrak{S}}$ is in $f[B]$, and so the “real" value range of $\mathfrak{M}^{\mathfrak{S}}$ is just $f[B]$. Since f is a monomorphism, B and $f[B]$ are isomorphic to each other, and hence $\mathfrak{A}$ and $\mathfrak{M}^{\mathfrak{S}}$ are isomorphic, and therefore $\mathfrak{A}$ is equivalent to a supervaluation model. □

Corollary 3.3.1 Let B be an atomic complete Boolean algebra. Any B-valued true identity model is equivalent to a supervaluation model.

The duality we established above shows that Boolean-valued models generalizes supervaluation models in two aspects. First, Boolean-valued models allow identity clauses to take intermediate truth values, whereas supervaluation models require true identity. Second, Boolean-valued models allow the value range of a model to be any complete Boolean algebra, whereas supervaluation models require powerset algebras (or those embeddable in a powerset algebra in a complete way).

4 Boolean Valuations

Thanks to Rasiowa and Sikorski, Boolean-valued models are known to be sound and complete with respect to first-order theories, in the following sense:[12]

Definition 4.1 Let T be a theory in a first order language $\mathscr{L}$. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is a model of T iff for any $\phi \in T$, $[\![\phi]\!]^{\mathfrak{A}} = 1_B$.
Let ϕ be a sentence in $\mathscr{L}$. ϕ is a Boolean-consequence of T, in symbols, $T \models_B \phi$ iff for any Boolean valued model $\mathfrak{A}$, if $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ is a model of ϕ.

Theorem 4.1 (Rasiowa, Sikorski) Let T be a theory and ϕ be a sentence in $\mathscr{L}$. $T \models_B \phi$ if and only if $T \vdash \phi$.

As corollaries:[13]

Corollary 4.1.1 1. Let ϕ be a tautology. In any Boolean valued model $\mathfrak{A}$, $[\![\phi]\!]^{\mathfrak{A}} = 1$.

[12] For a proof of Theorem 4.1, see [15].

[13] It is important to note that the results in Corollary 4.1.1 are essentially just variations of similar results in [15] and hence are due to Rasiowa and Sikorski, not the author.

2. Let T be a theory in $\mathscr{L}$. T is consistent iff for any complete Boolean Algebra B, T has a B-valued model.
3. For any complete Boolean Algebra B, T has a B-valued model iff every finite subset of T has a B-valued model.

When there are only two truth values, the notion of “theory" is sufficient for describing the relationship between models and sentences. Given a two-valued model of a language $\mathscr{L}$, the set of all sentences of $\mathscr{L}$ that are true in the model forms a complete theory in $\mathscr{L}$. This theory decides the value of all sentences of $\mathscr{L}$ in the model: if ϕ is a member of the theory, then ϕ has value 1 in the model, and if ϕ is not a member of the theory, then ϕ has value 0 in the model. This theory, in a certain sense, provides a full description of the model given that our expressive power is limited to $\mathscr{L}$.
The situation is different, however, when we allow more than two truth values. Given a B-valued model of $\mathscr{L}$ where B is a proper extension of 2, the theory in $\mathscr{L}$ that consists of all sentences of $\mathscr{L}$ that are true in the model no longer decides the value of all sentences of $\mathscr{L}$ in the model. A simple example to illustrate this point is as follows: Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two B-valued models of $\mathscr{L}$, where B is the four element Boolean algebra $\{0, p, -p, 1\}$ and $\mathscr{L}$ is the language $\{P, c\}$ where P is a unary predicate and c is a constant. Let $A = \{a\}$ and $A' = \{a'\}$. Let $[\![c]\!]^{\mathfrak{A}} = a$ and $[\![c]\!]^{\mathfrak{A}'} = a'$. Let $[\![P]\!]^{\mathfrak{A}}(a) = p$ and $[\![P]\!]^{\mathfrak{A}'}(a') = -p$. Then it is easy to see that the set of sentences of $\mathscr{L}$ that have value 1 in $\mathfrak{A}$ is the same as the set of sentences of $\mathscr{L}$ that have value 1 in $\mathfrak{A}'$. But obviously not all sentences of $\mathscr{L}$ have the same value in $\mathfrak{A}$ and $\mathfrak{A}'$.
This result is hardly surprising. Knowing which sentences have the top value only allows us to know the values of those sentences that have extreme values. When we only have two values, this amounts to knowing the value of every sentence. But whence we have more than two values, knowing the values of those that have extreme values is not enough: we still need to know the values of those that have intermediate values. And the latter is simply not decided by the former.
Therefore, in a Boolean-valued setting, we need a notion stronger than the notion of “theory", one that is sufficiently strong to fulfill the kind of jobs that the notion of “theory" does in the setting of two-valued models: one that is able to, for example, provide a full description of a model that decides the value of every sentence in the model. A natural candidate, as we will introduce right now, is the notion of “Boolean-valuations".

Definition 4.2 Let B be a complete Boolean algebra. Let $\mathscr{L}$ be a first order language. A Boolean-valuation S^B in $\mathscr{L}$ is a set of pairs of the form $\langle \phi, p\rangle$ such that ϕ is a sentence of $\mathscr{L}$ and p is an element of B. We say that B is the value range of the Boolean valuation S^B, or that S^B is a B-valuation.

Definition 4.3 Let S^B be a B-valuation of $\mathscr{L}$. Let $\mathfrak{A}$ be a B'-valued model of $\mathscr{L}$. $\mathfrak{A}$ is a model of S^B iff B is a complete subalgebra of B' and for any sentence $\phi \in \mathscr{L}$, for any $p \in B$, if $\langle \phi, p\rangle \in S^B$, then $[\![\phi]\!]^{\mathfrak{A}} = p$.

Intuitively, a Boolean-valuation assigns values of a Boolean algebra to certain sentences of a language. When a pair $\langle \phi, p\rangle$ is in the Boolean-valuation S^B, we can think of the Boolean-valuation “says" that the sentence ϕ has value p. If a model $\mathfrak{A}$ is a model of S^B, then figuratively, what S^B says about those sentences that are mentioned in S^B is what actually is the case in $\mathfrak{A}$. We can already see why the notion of Boolean-valuations will be useful for our purpose: a full description of a Boolean-valued model with respect to a particular language, intuitively, is simply an assignment of values to all the sentences in the language. But the latter, from a set-theoretic perspective, is just a collection of sentence-value pairs, which is simply a Boolean-valuation given our definition.

Also, theories, in a natural sense, can be understood as special cases of Boolean-valuations. Roughly, a theory T is a Boolean valuation $T^B = \{\langle \phi, 1\rangle \mid \phi \in T\}$. A model $\mathfrak{A}$ is a model of T just in case $\mathfrak{A}$ is a model of T^B. The notion of "Boolean-valuation" is a natural generalization of the notion of "theory", in the context of Boolean valued models.

An important property of theories is consistency. Consistent theories, as we have seen, precisely correspond to theories that have Boolean valued models. This is a nice synergy between syntax and semantics. But what about Boolean-valuations? What does it mean for a Boolean-valuation to be "consistent"? Are consistent Boolean-valuations precisely those that have models? These are the questions that we will answer for the rest of the section.

Definition 4.4 Let S^B be a Boolean-valuation of $\mathscr{L}$. Let $h : B \to 2$ be a homomorphism. S^B_h is the following set of sentences: for any $\phi \in \mathscr{L}$, any $p \in B$,

1. If $\langle \phi, p\rangle \in S^B$ and $h(p) = 1$, then $\phi \in S^B_h$.
2. If $\langle \phi, p\rangle \in S^B$ and $h(p) = 0$, then $\neg\phi \in S^B_h$.
3. Nothing else is in S^B_h.

Definition 4.5 A Boolean-valuation S^B is consistent if and only if for any homomorphism $h : B \to 2$, S^B_h is a consistent theory.

Consistency of Boolean-valuations is thus defined in terms of consistency of theories. Let T be a theory and let T^B be the Boolean-valuation $\{\langle \phi, 1\rangle \mid \phi \in T\}$. It follows straightforwardly from Definitions 4.4 and 4.5 that T is consistent just in case T^B is consistent in the sense of Definition 4.5, as every homomorphism takes 1_B to 1_2.

The major result of this section is that consistent Boolean-valuations are precisely those that have models. To reach that result, though, we will have to prove a series of subsidiary theorems first, which are also interesting on their own. In the following, whenever we mention a Boolean-valuation, we always assume that it is a Boolean-valuation of the language $\mathscr{L}$. Also, occasionally, we will call a Boolean-valuation S^B a B-valuation.

Definition 4.6 A Boolean-valuation S'^B is a sub-valuation of S^B if and only if $S'^B \subseteq S^B$ and the value range of S'^B is the same as that of S^B.

Theorem 4.2 If a Boolean-valuation S^B is consistent , then every sub-valuation of S^B is consistent.

Proof Let S'^B be a sub-valuation of S^B . Then for every homomorphism $h : B \to 2$, $S'^B_h \subseteq S^B_h$. If S'^B is inconsistent, then S'^B_h is inconsistent for some homomorphism h, and then S^B_h will be inconsistent. $\square$

Proposition 4.1 Let S^B be a Boolean-valuation and let $h : B \to 2$ be a homomorphism. For any finite subset $\Delta \subseteq S^B_h$, for some finite sub-valuation S'^B of S^B, $S'^B_h = \Delta$.

Theorem 4.3 A Boolean-valuation S^B is consistent if and only if every finite sub-valuation of S^B is consistent.

Proof The direction from left to right follows directly from Theorem 4.2.
For the other direction, let S^B be an inconsistent B-valuation. Then for some homomorphism $h : B \to 2$, S^B_h is inconsistent. Hence some finite subset T of S^B_h is inconsistent. By Proposition 4.1, for some finite sub-valuation T^B of S^B, $T^B_h = T$. Hence T^B_h is inconsistent. Hence T^B is inconsistent. □

Theorem 4.4 Let S^B be a consistent B-valuation. For any sentence $\psi \in \mathscr{L}$, for some $r \in B$, $S^B \cup \{\langle \psi, r\rangle\}$ is consistent.

Proof Let $X = \{h : B \to 2 \mid h \text{ is a homomorphism}\}$.
Let $K = \{\Delta^\beta \mid \Delta^\beta \text{ is a finite sub-valuation of } S^B\}$. Enumerate K by α where $\alpha = |K|$. For each $\beta < \alpha$, Δ^β is a finite sub-valuation of S^B, and $S^B = \bigcup_{\beta<\alpha} \Delta^\beta$.
For any $\beta < \alpha, h \in X$, we form Δ^β_h according to Definition 4.4. For any $\beta < \alpha, h \in X$, $\Delta^\beta_h \subseteq S^B_h$. Also for any $h \in X$, $\{\Delta^\beta_h \mid \beta < \alpha\} = \{\Delta \mid \Delta \text{ is a finite subset of } S^B_h\}$.
Fix an $\beta < \alpha$. Let $\Delta^\beta = \{\langle \phi_1, p_1\rangle, ..., \langle \phi_k, p_k\rangle\}$ for some $k < \omega$. For any $h \in X$, let $q^h_\beta = q_1 \sqcap ... \sqcap q_k$, where for any $1 \leqslant i \leqslant k$, $q_i = p_i$ if $h(p_i) = 1$, and $q_i = -p_i$ if $h(p_i) = 0$.
To continue with the proof we need to prove two claims.

Claim 4.4.1 For any $\beta < \alpha, h \in X, h(q^h_\beta) = 1$.

Proof of the Claim Let $q^h_\beta = q_1 \sqcap ... \sqcap q_k$ as defined above. Then for any $1 \leqslant i \leqslant k$, $h(q_i) = 1$. Hence $h(q^h_\beta) = 1$.

Let $J^+_\beta = \{h_j \in X \mid \Delta^\beta_{h_j} \vdash \psi\}$ and $J^-_\beta = \{h_k \in X \mid \Delta^\beta_{h_k} \vdash \neg\psi\}$.
Let $q^+_\beta = \bigsqcup_{h_j \in J^+_\beta} q^{h_j}_\beta$ and $q^-_\beta = \bigsqcup_{h_k \in J^-_\beta} q^{h_k}_\beta$.

Claim 4.4.2 For some $r \in B$, $r \geqslant \bigsqcup_{\beta<\alpha} q^+_\beta$ and $-r \geqslant \bigsqcup_{\beta<\alpha} q^-_\beta$.

Proof of the Claim We only need to show that

$$\bigsqcup_{\beta<\alpha} q^+_\beta \sqcap \bigsqcup_{\beta<\alpha} q^-_\beta = 0$$

By infinite distribution, this is equivalent to

$$\bigsqcup_{\beta,\gamma<\alpha} (q^+_\beta \sqcap q^-_\gamma) = 0$$

That is, for any $\beta, \gamma < \alpha$, $q_\beta^+ \sqcap q_\gamma^- = 0$, i.e.

$$\bigsqcup_{h_j \in J_\beta^+} q_\beta^{h_j} \sqcap \bigsqcup_{h_k \in J_\gamma^-} q_\gamma^{h_k} = 0$$

Again by infinite distribution, this is equivalent to

$$\bigsqcup_{h_j \in J_\beta^+} \bigsqcup_{h_k \in J_\gamma^-} (q_\beta^{h_j} \sqcap q_\gamma^{h_k}) = 0$$

That is, for any $h_j \in J_\beta^+$, any $h_k \in J_\gamma^-$, $q_\beta^{h_j} \sqcap q_\gamma^{h_k} = 0$.
Suppose not, then for some $h_j \in J_\beta^+$, $h_k \in J_\gamma^-$, for some $p \neq 0 \in B$, $q_\beta^{h_j} \sqcap q_\gamma^{h_k} = p$.
Since $p \neq 0$, there is some $h \in X$ such that $h(p) = 1$. Hence $h(q_\beta^{h_j}) = 1$, $h(q_\gamma^{h_k}) = 1$.
But by definition of $q_\beta^{h_j}$, then, for any p_i such that some pair of the form $\langle \phi_i, p_i \rangle \in \Delta^\beta$, if $h_j(p_i) = 1$, then $q_\beta^{h_j} \leqslant p_i$, and hence $h(p_i) = 1$. And similarly, if $h_j(p_i) = 0$, then $q_\beta^{h_j} \leqslant -p_i$, and hence $h(-p_i) = 1$, $h(p_i) = 0$.
Hence for any p_i such that some pair of the form $\langle \phi_i, p_i \rangle \in \Delta^\beta$, $h_j(p_i) = h(p_i)$.
Hence by Definition 4.4, $\Delta_{h_j}^\beta = \Delta_h^\beta$. Similarly, $\Delta_{h_k}^\gamma = \Delta_h^\gamma$
But since $h_j \in J_\beta^+$, $\Delta_{h_j}^\beta \vdash \psi$; and since $h_k \in J_\gamma^-$, $\Delta_{h_k}^\gamma \vdash \neg\psi$. Hence $\Delta_h^\beta \vdash \psi$, $\Delta_h^\gamma \vdash \neg\psi$.
But $\Delta_h^\beta \subseteq S_h^B$, $\Delta_h^\gamma \subseteq S_h^B$. Hence $S_h^B \vdash \psi \wedge \neg\psi$. Hence S_h^B is inconsistent. But this is a contradiction as S^B is assumed to be consistent.

Pick an $r \in B$ that witnesses Claim 4.4.2. Finally, we will show that $S^B \cup \{\langle \psi, r \rangle\}$ is consistent.
Suppose it is not consistent. Then for some $h \in X$, one of the two following situations holds:

(a) $h(r) = 1$ and $S_h^B \cup \{\psi\}$ is inconsistent.
(b) $h(r) = 0$ and $S_h^B \cup \{\neg\psi\}$ is inconsistent.

We will show that both (a) and (b) lead to contradiction.
Assume (a). Since $S_h^B \cup \{\psi\}$ is inconsistent, $S_h^B \vdash \neg\psi$. Hence for some $\beta < \alpha$, $\Delta_h^\beta \vdash \neg\psi$. Hence $h \in J_\beta^-$.
Hence $-r \geqslant \bigsqcup_{\gamma < \alpha} q_\gamma^- \geqslant q_\beta^- = \bigsqcup_{h_k \in J_\beta^-} q_\beta^{h_k} \geqslant q_\beta^h$.
But by Claim 4.4.1, $h(q_\beta^h) = 1$. Hence $h(-r) = 1$, $h(r) = 0$. Contradiction.
Assume (b). Since $S_h^B \cup \{\neg\psi\}$ is inconsistent, $S_h^B \vdash \psi$. Hence for some $\beta < \alpha$, $\Delta_h^\beta \vdash \psi$. Hence $h \in J_\beta^+$.
Hence $r \geqslant \bigsqcup_{\gamma < \alpha} q_\gamma^+ \geqslant q_\beta^+ = \bigsqcup_{h_j \in J_\beta^+} q_\beta^{h_j} \geqslant q_\beta^h$.

But by Claim 4.4.1, $h(q_\beta^h) = 1$. Hence $h(r) = 1$. Contradiction.

Definition 4.7 A Boolean-valuation S^B is maximal if and only if for every sentence ϕ, there is some $p \in B$ such that $\langle \phi, p \rangle \in S^B$.

Theorem 4.5 Every consistent Boolean-valuation is contained in some maximal consistent Boolean-valuation.

Proof Let S^B be a consistent B-valuation. Let $D = \{\langle \phi, p \rangle \mid \phi$ is a sentence of $\mathscr{L}$, $p \in B\}$. We use the Axiom of Choice to arrange all the pairs in D in a list:

$$\langle \phi_0, p_0 \rangle, \langle \phi_1, p_1 \rangle, \ldots, \langle \phi_\alpha, p_\alpha \rangle, \ldots \qquad \alpha < |D|$$

such that the list associates in a one-one fashion an ordinal with each pair.
We shall form an increasing chain of consistent B-valuations:

$$S^B = S_0^B \subseteq S_1^B \subseteq \ldots \subseteq S_\alpha^B \subseteq \ldots \qquad \alpha < |D|$$

If $S^B \cup \{\langle \phi_0, p_0 \rangle\}$ is consistent, define $S_1^B = S^B \cup \{\langle \phi_0, p_0 \rangle\}$. Otherwise, define $S_1^B = S^B$.
At the α-th stage, if α is a successor ordinal, define

$$\begin{cases} S_\alpha^B = S_{\alpha-1}^B \cup \{\langle \phi_{\alpha-1}, p_{\alpha-1} \rangle\} & \text{if } S_{\alpha-1}^B \cup \{\langle \phi_{\alpha-1}, p_{\alpha-1} \rangle\} \text{ is consistent} \\ S_\alpha^B = S_{\alpha-1}^B & \text{if otherwise} \end{cases}$$

If α is a limit ordinal, define $S_\alpha^B = \bigcup\limits_{\beta < \alpha} S_\beta^B$. Let T^B be the union of all the S_α^B's.

Claim 4.5.1 T^B is a consistent B-valuation.

Proof of the Claim Suppose not. Then for some homomorphism $h : B \to 2$, T_h^B is inconsistent. Then for some finite subset $\{\psi_1, \psi_2, \ldots, \psi_k\} \subseteq T_h^B$, $\{\psi_1, \psi_2, \ldots, \psi_k\}$ is inconsistent. By Proposition 4.1, for some finite sub-valuation Δ^B of T^B, $\Delta_h^B = \{\psi_1, \psi_2, \ldots, \psi_k\}$. Hence Δ^B is inconsistent. But since Δ^B is finite, for some $\alpha < |D|$, $\Delta^B \subseteq S_\alpha^B$. But then S_α^B is inconsistent. Contradiction. □

Claim 4.5.2 T^B is maximal.

Proof of the Claim Let ϕ be a sentence of $\mathscr{L}$. By Theorem 4.4, for some $p \in B$, $T^B \cup \{\langle \phi, p \rangle\}$ is consistent. But then $\{\langle \phi, p \rangle\}$ will be added to T^B at the stage when it is enumerated.

Hence S^B is contained in a maximal consistent B-valuation, namely T^B. □

When S^B is a consistent B-valuation, it is easy to show that for any sentence ϕ, for any $p, q \in B$, if $\langle \phi, p \rangle$ and $\langle \phi, q \rangle$ are both in S^B, then $p = q$. This is because if otherwise, then there is some homomorphism $h : B \to 2$ such that $h(p) \neq h(q)$, and hence both ϕ and $\neg\phi$ will be in S_h^B, making S^B inconsistent. Hence, in the following,

when the context is clear, we will use the term $[\![\phi]\!]^S$ to denote the unique p such that $\langle\phi, p\rangle \in S^B$.
With the help of Theorems 4.4 and 4.5 we are finally able to prove the completeness theorem on Boolean-valuations.

Theorem 4.6 Let S^B be a consistent Boolean-valuation of $\mathscr{L}$. Then S^B has a B-valued model that is witnessing.

Proof Let $X = \{h : B \to 2 \mid h$ is a homomorphism$\}$.
Let S^B be a consistent Boolean-valuation in $\mathscr{L}$. Let C be a set of new constants (not appearing in $\mathscr{L}$) with the same cardinality of $\mathscr{L}$. Let $\mathscr{L}' = \mathscr{L} \cup C$.
Arrange all formulas with one free variable in $\mathscr{L}'$ into a list:

$$\phi_0, \phi_1, ..., \phi_\alpha, ... \qquad \alpha < |\mathscr{L}|$$

We now define an increasing sequence of B-valuations of $\mathscr{L}'$:

$$S^B = S^B_0 \subseteq S^B_1 \subseteq ... \subseteq S^B_\alpha \subseteq ... \qquad \alpha < |\mathscr{L}|$$

and a sequence $d_1, ..., d_\alpha, ..., \alpha < |\mathscr{L}|$, of constants from C, in the following way:
Let $\alpha + 1$ be a successor ordinal, then define $S^B_{\alpha+1}$ as follows: first add to S^B_α a pair of the form $\langle\exists v_\alpha \phi_\alpha(v_\alpha), p\rangle$ such that $S^B_\alpha \cup \{\langle\exists v_\alpha \phi_\alpha(v_\alpha), p\rangle\}$ is consistent. Theorem 4.4 guarantees the existence of such a pair. Then, we let d_α be the first constant in C that has not appeared in $S^B_\alpha \cup \{\langle\exists v_\alpha \phi_\alpha(v_\alpha), p\rangle\}$. The existence of a new constant is guaranteed by the cardinality of C. Then, we add to S^B_α the pair $\langle\phi_\alpha(d_\alpha), p\rangle$.
If α is a limit ordinal, then let $S^B_\alpha = \bigcup_{\beta<\alpha} S^B_\beta$.

Claim 4.6.1 S^B_α is consistent for any $\alpha < |\mathscr{L}|$.

Proof of the Claim We use transfinite induction. We first show that at the successor stage, if S^B_α is consistent, then $S^B_{\alpha+1} = S^B_\alpha \cup \{\langle\exists v_\alpha \phi_\alpha(v_\alpha), p\rangle, \langle\phi_\alpha(d_\alpha), p\rangle\}$ is consistent. Suppose not. Then for some $h \in X$, $(S^B_{\alpha+1})_h$ is inconsistent. There are two situations:

(a) $h(p) = 1$. Then $(S^B_\alpha)_h \cup \{\exists v_\alpha \phi_\alpha(v_\alpha), \phi_\alpha(d_\alpha)\}$ is inconsistent. Then $(S^B_\alpha)_h \cup \{\exists v_\alpha \phi_\alpha(v_\alpha)\} \vdash \neg\phi_\alpha(d_\alpha)$. Since d_α does not appear on the left hand side, $(S^B_\alpha)_h \cup \{\exists v_\alpha \phi_\alpha(v_\alpha)\} \vdash \forall v_\alpha \neg\phi_\alpha(v_\alpha)$. But then $(S^B_\alpha)_h \cup \{\exists v_\alpha \phi_\alpha(v_\alpha)\}$ is inconsistent, contradicting our choice of p.
(b) $h(p) = 0$. Then $(S^B_\alpha)_h \cup \{\neg\exists v_\alpha \phi_\alpha(v_\alpha), \neg\phi_\alpha(d_\alpha)\}$ is inconsistent. Then $(S^B_\alpha)_h \cup \{\neg\exists v_\alpha \phi_\alpha(v_\alpha)\} \vdash \phi_\alpha(d_\alpha)$. Since d_α does not appear on the left hand side, $(S^B_\alpha)_h \cup \{\neg\exists v_\alpha \phi_\alpha(v_\alpha)\} \vdash \forall v_\alpha \phi_\alpha(v_\alpha)$. But then $(S^B_\alpha)_h \cup \{\neg\exists v_\alpha \phi_\alpha(v_\alpha)\}$ is inconsistent, contradicting our choice of p.

At the limit stage, if S^B_α is inconsistent, then by Theorem 4.3, a finite sub-valuation of S^B_α is inconsistent, meaning that some S^B_β is inconsistent, where $\beta < \alpha$, contradicting the inductive hypothesis. □

Let $T'^B = \bigcup_{\alpha<|\mathscr{L}|} S_\alpha^B$. T'^B is consistent, for the same reason why S_α^B is consistent when α is a limit ordinal. Since T'^B is a consistent, by Theorem 4.5 it is contained in some maximal consistent B-valuation of $\mathscr{L}'$. Let T^B be such a B-valuation.
Let $A = C$. We will construct a B-valued model $\mathfrak{A}$ of $\mathscr{L}'$ with universe A/C:

1. Let c be a constant in $\mathscr{L}'$. Then $[\![c]\!]^{\mathfrak{A}} = d_\alpha$ such that $[\![c = d_\alpha]\!]^T = [\![\exists v(v = c)]\!]^T$. (If there is more than one $d_\alpha \in A$ that satisfies this, then just pick a random one.)
2. Let P be an n-nary relation. For any $\langle c_1, ..., c_n\rangle \in A^n$, let $[\![P(c_1, ..., c_n)]\!]^{\mathfrak{A}} = [\![P(c_1, ..., c_n)]\!]^T$.
3. For the identity symbol, for any $c_\alpha, c_\beta \in A$, let $[\![c_\alpha = c_\beta]\!]^{\mathfrak{A}} = [\![c_\alpha = c_\beta]\!]^T$.

Claim 4.6.2 $\mathfrak{A}$ is a B-valued model.

Proof of the Claim For any $d_\alpha, d_\beta, d_\gamma \in A$,

(1) $[\![d_\alpha = d_\alpha]\!]^{\mathfrak{A}} = 1$.
Suppose not. Then for some $h \in X$, $h([\![d_\alpha = d_\alpha]\!]^{\mathfrak{A}}) = 0$. Then $d_\alpha \neq d_\alpha \in T_h^B$, making T_h^B inconsistent.

(2) $[\![d_\alpha = d_\beta]\!]^{\mathfrak{A}} = [\![d_\beta = d_\alpha]\!]^{\mathfrak{A}}$
Suppose not. Then for some $h \in X$, $h([\![d_\alpha = d_\beta]\!]^{\mathfrak{A}}) \neq h([\![d_\beta = d_\alpha]\!]^{\mathfrak{A}})$. Then (without loss of generality) $d_\alpha = d_\beta \in T_h^B$ and $d_\beta \neq d_\alpha \in T_h^B$, making T_h^B inconsistent.

(3) $[\![d_\alpha = d_\beta]\!]^{\mathfrak{A}} \sqcap [\![d_\beta = d_\gamma]\!]^{\mathfrak{A}} \leqslant [\![d_\alpha = d_\gamma]\!]^{\mathfrak{A}}$
Suppose not. Then for some $h \in X$, $h([\![d_\alpha = d_\gamma]\!]^{\mathfrak{A}}) = 0$ but $h([\![d_\alpha = d_\beta]\!]^{\mathfrak{A}} \sqcap [\![d_\beta = d_\gamma]\!]^{\mathfrak{A}}) = 1$. Hence $h([\![d_\alpha = d_\beta]\!]^{\mathfrak{A}}) = 1$ and $h([\![d_\beta = d_\gamma]\!]^{\mathfrak{A}})) = 1$. Hence $d_\alpha = d_\beta, d_\beta = d_\gamma \in T_h^B$ but $d_\alpha \neq d_\gamma \in T_h^B$, making T_h^B inconsistent.

(4) For any n-nary relation P, for any $\langle c_1, ..., c_n\rangle, \langle c_1', ..., c_n'\rangle \in A^n$,

$$[\![P(c_1, ..., c_n)]\!]^{\mathfrak{A}} \sqcap (\prod_{1\leqslant i\leqslant n} [\![c_i = c_i']\!]^{\mathfrak{A}}) \leqslant [\![P(c_1', ..., c_n')]\!]^{\mathfrak{A}}$$

For simplicity we only prove for the case when $n = 1$. The proofs for the cases when $n > 1$ are very similar.
Suppose not. Then for some $h \in X$, $h([\![P(c_1')]\!]^{\mathfrak{A}}) = 0$ but $h([\![c_1 = c_1']\!]^{\mathfrak{A}} \sqcap [\![P(c_1)]\!]^{\mathfrak{A}}) = 1$. Hence $h([\![c_1 = c_1']\!]^{\mathfrak{A}}) = 1$ and $h([\![P(c_1)]\!]^{\mathfrak{A}}) = 1$. Hence $c_1 = c_1'$, $P(c_1) \in T_h^B$ but $\neg P(c_1') \in T_h^B$, making T_h^B inconsistent. □

Finally we will show that $\mathfrak{A}$ is a model of T^B, i.e. for any sentence ϕ of $\mathscr{L}'$, $[\![\phi]\!]^{\mathfrak{A}} = [\![\phi]\!]^T$.
We prove by induction on the complexity of ϕ.
Atomic cases:

(a) $[\![c = c']\!]^{\mathfrak{A}} = [\![d_\alpha = d_\beta]\!]^T$ where $[\![c = d_\alpha]\!]^T = [\![\exists v(c = v)]\!]^T = 1$ and $[\![c' = d_\beta]\!]^T = [\![\exists v(c' = v)]\!]^T = 1$.
We just need to show that $p = q$ when $p = [\![d_\alpha = d_\beta]\!]^T$ and $q = [\![c = c']\!]^T$.
Suppose not. Then for some $h \in X$, $h(p) \neq h(q)$. Hence (WLOG) $d_\alpha = d_\beta \in T_h^B$, $c \neq c' \in T_h^B$. But $c = d_\alpha$, $c' = d_\beta \in T_h^B$. T_h^B is inconsistent. Contradiction.

(b) For the atomic cases of relations, again, we just show it for unary relations. The cases of other n-nary relations where $n > 1$ are very similar.

$[\![P(c)]\!]^{\mathfrak{A}} = [\![P(d_\alpha)]\!]^T$ where $[\![c = d_\alpha]\!]^T = [\![\exists v_i(c = v_i)]\!]^T = 1$.
We just need to show that $p = q$ when $p = [\![P(d_\alpha)]\!]^T$ and $q = [\![P(c)]\!]^T$.
Suppose not. Then for some $h \in X$, $h(p) \neq h(q)$. Hence (WLOG) $P(d_\alpha) \in T_h^B$, $\neg P(c) \in T_h^B$. But $c = d_\alpha \in T_h^B$. T_h^B is inconsistent. Contradiction.

Inductive cases:

(a) $\phi = \neg\psi$.

$$[\![\phi]\!]^{\mathfrak{A}} = [\![\neg\psi]\!]^{\mathfrak{A}} = -[\![\psi]\!]^{\mathfrak{A}} = -[\![\psi]\!]^T = [\![\neg\psi]\!]^T$$

The last equation holds for the following reasons. Suppose not, and suppose $[\![\psi]\!]^T = p$ and $[\![\neg\psi]\!]^T = q \neq -p$. Then for some $h \in X$, $h(-p) \neq h(q)$. WLOG we can assume $h(-p) = 1$ and $h(q) = 0$. Then $h(p) = 0$. Then $\neg\psi \in T_h^B$ and $\neg\neg\psi \in T_h^B$, making T_h^B inconsistent. Contradiction.

(b) $\phi = \psi_1 \wedge \psi_2$.

$$[\![\psi_1 \wedge \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \sqcap [\![\psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^T \sqcap [\![\psi_2]\!]^T = [\![\psi_1 \wedge \psi_2]\!]^T$$

The last equation holds for the following reasons. Suppose not, and suppose $[\![\psi_1]\!]^T \sqcap [\![\psi_2]\!]^T = p \neq q = [\![\psi_1 \wedge \psi_2]\!]^T$. Then for some $h \in X$, $h(p) = 1$ and $h(q) = 0$, or $h(p) = 0$ and $h(q) = 1$. Suppose $h(p) = 1$ and $h(q) = 0$. Then $\psi_1, \psi_2 \in T_h^B$, but $\neg(\psi_1 \wedge \psi_2) \in T_h^B$, making T_h^B inconsistent. On the other hand, suppose $h(p) = 0$ and $h(q) = 1$. Then $\psi_1 \wedge \psi_2 \in T_h^B$. Then both $h([\![\psi_1]\!]^T)$ and $h([\![\psi_2]\!]^T)$ have to be 1, as otherwise $\neg\psi_1$ or $\neg\psi_2$ would be in T_h^B, which would make T_h^B inconsistent. But then $h([\![\psi_1]\!]^T \sqcap [\![\psi_2]\!]^T) = h(p)$ has to be 1. Contradiction.

(c) $\phi = \psi_1 \vee \psi_2$.

$$[\![\psi_1 \vee \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \sqcup [\![\psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^T \sqcup [\![\psi_2]\!]^T = [\![\psi_1 \vee \psi_2]\!]^T$$

The last equation holds for the following reasons. Suppose not, and suppose $[\![\psi_1]\!]^T \sqcup [\![\psi_2]\!]^T = p \neq q = [\![\psi_1 \vee \psi_2]\!]^T$. Then for some $h \in X$, $h(p) = 1$ and $h(q) = 0$, or $h(p) = 0$ and $h(q) = 1$. Suppose $h(p) = 1$ and $h(q) = 0$. Then $\neg(\psi_1 \vee \psi_2) \in T_h^B$, and hence both $h([\![\psi_1]\!]^T)$ and $h([\![\psi_2]\!]^T)$ have to be 0 as otherwise ψ_1 or ψ_2 would be in T_h^B, which would make T_h^B inconsistent. But then $h([\![\psi_1]\!]^T \sqcup [\![\psi_2]\!]^T) = h(p)$ has to be 0. Contradiction. On the other hand, suppose $h(p) = 0$ and $h(q) = 1$. Then $h([\![\psi_1]\!]^T) = 0$, $h([\![\psi_2]\!]^T) = 0$. Hence $\neg\psi_1, \neg\psi_2 \in T_h^B$, but $\psi_1 \vee \psi_2 \in T_h^B$, making T_h^B inconsistent.

(d) $\phi = \exists v\psi(v)$.
Let $\theta(v)$ be any formula with only v free. Then it is easy to show that for any $d \in A$, $[\![\theta(v)]\!]^{\mathfrak{A}}[d] = [\![\theta(d)]\!]^{\mathfrak{A}}$, as $[\![d]\!]^{\mathfrak{A}}$ is some $d_\alpha \in A$ such that $[\![d = d_\alpha]\!]^{\mathfrak{A}} = 1$. Hence,

$$[\![\exists v\psi(v)]\!]^{\mathfrak{A}} = \bigsqcup_{d_\alpha \in A} [\![\psi(v)]\!]^{\mathfrak{A}}[d_\alpha] = \bigsqcup_{d_\alpha \in A} [\![\psi(d_\alpha)]\!]^{\mathfrak{A}} = \bigsqcup_{d_\alpha \in A} [\![\psi(d_\alpha)]\!]^T$$

We need to show that $\bigsqcup_{d_\alpha \in A} [\![\psi(d_\alpha)]\!]^T = [\![\exists v\psi(v)]\!]^T$.

For the $\leqslant$ direction: We just need to show that for any $d_\alpha \in A$, $[\![\psi(d_\alpha)]\!]^T \leqslant [\![\exists v\psi(v)]\!]^T$. Suppose not, and suppose for some $d_\alpha \in A$, $[\![\psi(d_\alpha)]\!]^T = p$ and $[\![\exists v\psi(v)]\!]^T = q$ and $p \nleqslant q$. Then $p \sqcap -q \neq 0$. Then for some $h \in X$, $h(p \sqcap -q) = 1$. Then $h(q) = 0$, and hence $\neg\exists v\psi(v) \in T_h^B$. But $\psi(d_\alpha) \in T_h^B$, making T_h^B inconsistent.

For the $\geqslant$ direction: by the setup of T'^B (hence of T^B), at some stage of the sequence (say, the αth stage), both $\langle \exists v_\alpha \psi(v_\alpha), p\rangle$ and $\langle \psi(d_\alpha), p\rangle$ are added to T'^B, for some $p \in B$. Hence for some $d_\alpha \in A$, $[\![\exists v\psi(v)]\!]^T = [\![\psi(d_\alpha)]\!]^T$.

Finally obviously $\mathfrak{A}$ is witnessing. □

Corollary 4.6.1 (Completeness) If S^B is consistent, then it has a model.

Theorem 4.7 (Soundness) If S^B has a model, then it is consistent.

Proof Let $\mathfrak{A}$ be a B'-valued model of S^B. Suppose S^B is inconsistent, then for some homomorphism $h : B \to 2$, S_h^B is inconsistent. Then, some finite subset $\Delta_h \subseteq S_h^B$ is inconsistent.

Let $\Delta_h = \{\phi_1, ..., \phi_n\}$. Let $\phi = \phi_1 \wedge ... \wedge \phi_n$. Clearly ϕ is a contradiction. Hence by Corollary 4.1.1, $[\![\phi]\!]^{\mathfrak{A}} = 0$.

Let $1 \leqslant i \leqslant n$. Consider ϕ_i. Since $\phi_i \in \Delta_h \subseteq S_h^B$, there are two possibilities:

(1) for some $p_i \in B$, $\langle \phi_i, p_i\rangle \in S^B$, and $h(p_i) = 1$;
(2) for some $p_i \in B$, $\langle \psi_i, p_i\rangle \in S^B$, and $h(p_i) = 0$, $\phi_i = \neg\psi_i$.

Suppose (1). Then since $\mathfrak{A}$ is a model of S^B, $[\![\phi_i]\!]^{\mathfrak{A}} = p_i$. $h([\![\phi_i]\!]^{\mathfrak{A}}) = h(p_i) = 1$.

Suppose (2). Then since $\mathfrak{A}$ is a model of S^B, $[\![\psi_i]\!]^{\mathfrak{A}} = p_i$. $[\![\phi_i]\!]^{\mathfrak{A}} = [\![\neg\psi_i]\!]^{\mathfrak{A}} = -p_i$. $h([\![\phi_i]\!]^{\mathfrak{A}}) = h(-p_i) = -h(p_i) = -0 = 1$.

In either case, $h([\![\phi_i]\!]^{\mathfrak{A}}) = 1$.

Hence $h([\![\phi]\!]^{\mathfrak{A}}) = h([\![\phi_1 \wedge ... \wedge \phi_n]\!]^{\mathfrak{A}}) = h([\![\phi_1]\!]^{\mathfrak{A}}) \sqcap ... \sqcap h([\![\phi_n]\!]^{\mathfrak{A}}) = 1 \sqcap ... \sqcap 1 = 1$.

Hence $h([\![\phi]\!]^{\mathfrak{A}}) \neq 0$. Contradiction. □

Corollary 4.7.1 A Boolean-valuation S^B is consistent if and only if it has a model.

Corollary 4.7.2 (Compactness) A Boolean-valuation S^B of $\mathscr{L}$ has a B-valued model if and only if every finite sub-valuation of S^B has a B-valued model.

Corollary 4.7.3 (Downward-Löwenheim) If a Boolean-valuation S^B of $\mathscr{L}$ has a B-valued model, then it has a witnessing B-valued model of size $\leqslant |\mathscr{L}|$.

5 Relationship Between Models

Two-valued models can stand in different relationships with one another: for example, a model can be isomorphic to another, a model can be a submodel of another, a model can be an elementary submodel of another, etc. These concepts are the cornerstone of the theory of two-valued models. The primary goal of this section is to generalize these concepts to Boolean-valued models.

5.1 Duplicate Resistant Models

Before we move on to generalize these concepts, there is one important complication that I have to resolve first, which will be relevant to our later purposes. Astute readers might have already noticed that the identity symbol is interpreted somewhat abnormally in the Boolean-valued models. The main abnormality, of course, is that a Boolean-valued model might "think" that two objects in its domain are identical to an intermediate degree between 0 and 1. We will talk more about identity in Boolean-valued models in Section 6. For current purposes, we will simply focus on the following minor yet interesting feature of Boolean-valued models: our definition of Boolean-valued models (Definition 2.1) allows there to be "duplicates" in the models - that is, two different objects a, b in the domain such that the value of $a = b$ is 1 in the model. The existence of duplicates in a model, in a natural sense, is both harmless and useless. To illustrate this point, we first introduce a new notion.

Definition 5.1 A B-valued mode $\mathfrak{A}$ of $\mathscr{L}$ is *duplicate resistant* just in case for any $a, b \in A$, if $[\![a = b]\!]^{\mathfrak{A}} = 1$, then a and b are the same element.

In other words, duplicate resistant models are those that disallow duplicates. The next results show that any Boolean-valued model is practically equivalent to a duplicate resistant model. But before that, we need an extra piece of definition.

Definition 5.2 Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let C be a complete Boolean algebra. Let $h : B \to C$ be a homomorphism. Then the C-valued *quotient model* $\mathfrak{A}^h$ of $\mathscr{L}$ is defined as follows:

1. Universe:
   Let $a_1, a_2 \in A$, define $a_1 \equiv_h a_2$ iff $h([\![a_1 = a_2]\!]^{\mathfrak{A}}) = 1_C$.
   It is easy to show that $\equiv_h$ is an equivalence relation on A^2, using Definition 2.1.
   Given $a_i \in A$, let $[a_i]_h = \{a_j \in A \mid a_i \equiv_h a_j\}$. Let the universe of $\mathfrak{A}^h$ be $A^h = \{[a_i]_h \mid a_i \in A\}$.
2. $[\![=]\!]^{\mathfrak{A}^h} : A^h \times A^h \to C$ is the function such that for any $[a_1]_h, [a_2]_h \in A^h$,
$$[\![[a_1]_h = [a_2]_h]\!]^{\mathfrak{A}^h} = h([\![a_1 = a_2]\!]^{\mathfrak{A}})$$
3. Let P be an n-ary relation in $\mathscr{L}$. $[\![P]\!]^{\mathfrak{A}^h} : (A^h)^n \to C$ is the function such that for any $\langle [a_1]_h, ..., [a_n]_h \rangle \in (A^h)^n$,
$$[\![P([a_1]_h, ..., [a_n]_h)]\!]^{\mathfrak{A}^h} = h([\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}})$$
   It is easy to show that $[\![=]\!]^{\mathfrak{A}^h}$ and $[\![P]\!]^{\mathfrak{A}^h}$ are well-defined.
4. Let c be a constant in $\mathscr{L}$. $[\![c]\!]^{\mathfrak{A}^h} = [[\![c]\!]^{\mathfrak{A}}]_h$.

Lemma 5.0.1 Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a complete homomorphism. Let x, x' be assignments on $\mathfrak{A}$ such that for any $v_i \in Var$, $x(v_i) \equiv_h x'(v_i)$. Then, for any formula ϕ of $\mathscr{L}$,
$$h([\![\phi]\!]^{\mathfrak{A}}[x]) = h([\![\phi]\!]^{\mathfrak{A}}[x'])$$

Proof By induction on the complexity of ϕ. □

Theorem 5.1 Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a complete homomorphism. Let $\mathfrak{A}^h$ be the C-valued quotient model as defined in Definition 5.2. Given $x : Var \to A^h$ an arbitrary assignment on $\mathfrak{A}^h$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any formula ϕ in $\mathscr{L}$,

$$[\![\phi]\!]^{\mathfrak{A}^h}[x] = h([\![\phi]\!]^{\mathfrak{A}}[y])$$

Proof By induction on the complexity of ϕ, with the help of Lemma 5.0.1. □

Theorem 5.2 (Generalized Łos' Theorem) Let $\mathfrak{A}$ be a witnessing B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a homomorphism. Let $\mathfrak{A}^h$ be the C-valued quotient model. Given $x : Var \to A^h$ an arbitrary assignment on $\mathfrak{A}^h$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any formula ϕ in $\mathscr{L}$,

$$[\![\phi]\!]^{\mathfrak{A}^h}[x] = h([\![\phi]\!]^{\mathfrak{A}}[y])$$

Proof See [21] or [19]. □

Definition 5.3 Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $h : B \to B$ be the identity function on B. The *duplicate resistant copy* of $\mathfrak{A}$, $\mathfrak{A}^d$, is the B-valued quotient model $\mathfrak{A}^h$ of $\mathscr{L}$.

Theorem 5.3 Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$, and let $\mathfrak{A}^d$ be its duplicate resistant copy, as defined in Definition 5.3. Given $x : Var \to A^d$ an arbitrary assignment on $\mathfrak{A}^d$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any formula ϕ,

$$[\![\phi]\!]^{\mathfrak{A}^d}[x] = [\![\phi]\!]^{\mathfrak{A}}[y]$$

Proof The proof is a straightforward application of Theorem 5.1, since the identity function $h : B \to B$ is a complete homomorphism. □

In other words, the value of any formula under some assignment x in the original model is the same as the value of the formula in the duplicate resistant copy, when we assign instead of objects equivalence classes of objects to the variables. As a consequence, all sentences have the same value in the duplicate resistant copy.

We have argued that the existence of duplicates is harmless and useless, from a technical point of view[14]. This is mostly true, except that the existence of duplicates creates some technical difficulty when we intend to generalize concepts like isomorphism. Consider a model $\mathfrak{A}$ with a finite domain and consider adding to $\mathfrak{A}$ a new object b such

[14] The reason why I do not block the existence of duplicates in the definition of Boolean-valued models, like in the case of two-valued models, is that the possibility of having duplicates might have interesting applications to certain philosophical issues. Models are relative to languages. And sometimes the language under concern might have limited expressive power in that it cannot distinguish between two potentially different objects. If we understand "=" as meaning "indistinguishable", then, we would want to allow there to be objects that are "duplicates" of each other, in the sense defined above.

that b is added as a duplicate of an original object a. Call the latter model $\mathfrak{A}'$. How is $\mathfrak{A}$ and $\mathfrak{A}'$ related? Intuitively, they should be practically the same. The addition of b is null in the sense that it makes no contribution to the evaluation of formulas. We would want our theory to indicate that the two models are isomorphic. Nevertheless, if we generalize the concept of isomorphism in the most straightforward way, $\mathfrak{A}$ and $\mathfrak{A}'$ will not be isomorphic. This is because, in the two-valued framework, an isomorphism between models has to be a bijection. But there is no bijection between A and A'.
One natural solution to resolve all these difficulties is to first define the notions of isomorphism, submodel, etc. on duplicate resistant models, in the most straightforward way, and then define isomorphism, etc. on arbitrary Boolean-valued models using the former. For example, we can define two Boolean-valued models as isomorphic just in case their duplicate resistant copies are isomorphic. This is going to be the method that we will adopt in the following subsections, as I believe that under this method we have the most natural and simple definitions for concepts like isomorphism. Alternative methods are available, of course: for example, we can give a definition of isomorphism under which isomorphisms do not have to be bijections. In the end, which method we adopt is more of a matter of taste than a matter of mathematical significance.

5.2 Isomorphism, Submodel, and Diagram

In this and the next two subsections, for reasons we have given in the previous subsection, we will assume all Boolean-valued models are duplicate resistant. Also, whenever we do not mention explicitly, we assume all models are models of a first-order language $\mathscr{L}$.

Definition 5.4 (Isomorphism) Let $\mathfrak{A}_1$ be a B_1-valued model and $\mathfrak{A}_2$ be a B_2-valued model. $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *isomorphic* iff there are functions $h : B_1 \to B_2$ and $f : A_1 \to A_2$ such that h is an isomorphism between Boolean algebras, and f is a bijection such that: (let t_i be a term)

1. For any $a_1, a_2 \in A_1$, $h([\![t_1 = t_2]\!]^{\mathfrak{A}_1})[a_1, a_2] = [\![t_1 = t_2]\!]^{\mathfrak{A}_2}[f(a_1), f(a_2)]$.
2. Let P be an n-nary predicate. For any $\langle a_1, ..., a_n\rangle \in A_1^n$, $h([\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n]) = [\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_2}[f(a_1), ..., f(a_n)]$.
3. Let c be a constant. $[\![c]\!]^{\mathfrak{A}_2} = f([\![c]\!]^{\mathfrak{A}_1})$.

In the rest of this section, for better readability, we will often identify a complete Boolean algebra with its isomorphic copies (and a Boolean value with its image under the isomorphism). It is routine to check that all the definitions and proofs will still work if we replace the value range of a Boolean-valued model with one of its isomorphic copy and interpretations of the symbols accordingly.

Definition 5.5 (Submodel) Let $\mathfrak{A}_1$ be a B_1-valued model and $\mathfrak{A}_2$ be a B_2-valued model. $\mathfrak{A}_1$ is a *submodel* of $\mathfrak{A}_2$ just in case: (let t_i be a term)

1. $A_1 \subseteq A_2$ and B_1 is (isomorphic to) a complete subalgebra of B_2.
2. For any $a_1, a_2 \in A_1$, $[\![t_1 = t_2]\!]^{\mathfrak{A}_1}[a_1, a_2] = [\![t_1 = t_2]\!]^{\mathfrak{A}_2}[a_1, a_2]$.[15]

[15] Here and in the rest of this definition, we are ignoring the difference between a Boolean value and its image under an isomorphism between Boolean algebras.

3. Let P be an n-nary predicate. For any $\langle a_1, ..., a_n\rangle \in A_1^n$, $[\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n] = [\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_2}[a_1, ..., a_n]$.
4. Let c be a constant. $[\![c]\!]^{\mathfrak{A}_2} = [\![c]\!]^{\mathfrak{A}_1}$.

Definition 5.6 (Diagram) Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $\mathscr{L}_{\mathfrak{A}} = \mathscr{L} \cup \{c_a \mid a \in A\}$, where $\{c_a \mid a \in A\}$ is a new set of constants, one for each $a \in A$. Expand $\mathfrak{A}$ to a model of $\mathscr{L}_{\mathfrak{A}}$ (call it $\mathfrak{A}^*$) such that for all $a \in A$, $[\![c_a]\!]^{\mathfrak{A}^*} = a$.
The *diagram* of $\mathfrak{A}$ is the B-valuation S^B which consists of and only of all the pairs of the form $\langle \phi, [\![\phi]\!]^{\mathfrak{A}^*}\rangle$ where ϕ is an atomic sentence or the negation of an atomic sentence of $\mathscr{L}_{\mathfrak{A}}$ and $[\![\phi]\!]^{\mathfrak{A}^*}$ is the value of ϕ in $\mathfrak{A}^*$.

Theorem 5.4 Let $\mathfrak{A}_1$ be a B_1-valued model and $\mathfrak{A}_2$ be a B_2-valued model. The followings are equivalent:

(1) $\mathfrak{A}_1$ is isomorphic to a submodel of $\mathfrak{A}_2$.
(2) $\mathfrak{A}_2$ can be expanded to a model of the diagram of $\mathfrak{A}_1$.

Proof (1) $\Rightarrow$ (2). WLOG we assume $\mathfrak{A}_1$ is isomorphic to $\mathfrak{A}_3$, which is a B_1-valued submodel of $\mathfrak{A}_2$ (so B_1 is a complete subalgebra of B_2), and the isomorphism is witnessed by $f : A_1 \to A_3 \subseteq A_2$. Expand $\mathfrak{A}_2$ to a model of $\mathscr{L}_{\mathfrak{A}_1}$ (call it $\mathfrak{A}'_2$) as follows: for any $a \in A_1$, let $[\![c_a]\!]^{\mathfrak{A}'_2} = f(a)$. It is routine to check that $\mathfrak{A}'_2$ is a model of the diagram of $\mathfrak{A}_1$, in the sense of Definition 4.3.

(2) $\Rightarrow$ (1). WLOG we assume B_1 is a complete subalgebra of B_2. Let $\mathfrak{A}'_2$ be an expansion of $\mathfrak{A}_2$ to the diagram of $\mathfrak{A}_1$. Construct $f : A_1 \to A_2$ as follows: for any $a \in A_1$, $f(a) = [\![c_a]\!]^{\mathfrak{A}'_2}$. Let $\mathfrak{A}_3$ be the submodel of $\mathfrak{A}_2$ whose domain is generated by $f[A_1]$.
We can show that the domain of $\mathfrak{A}_3$ is precisely $f[A_1]$. Let c be a constant in $\mathscr{L}$. And suppose $[\![c]\!]^{\mathfrak{A}_1} = a \in A_1$. Then $[\![c = c_a]\!]^{\mathfrak{A}_1^*} = 1$ and therefore $[\![c = c_a]\!]^{\mathfrak{A}'_2} = 1$. Since $\mathfrak{A}_2$ is duplicate resistant, $\mathfrak{A}'_2$ is also duplicate resistant. Hence $[\![c]\!]^{\mathfrak{A}'_2} = [\![c_a]\!]^{\mathfrak{A}'_2}$. Hence $[\![c]\!]^{\mathfrak{A}_3} = f(a) \in f[A_1]$.
We can easily show that $f : A_1 \to A_3$ is a bijection. Trivially it is surjective. Suppose $f(a_1) = f(a_2)$, then $[\![c_{a_1}]\!]^{\mathfrak{A}'_2} = [\![c_{a_2}]\!]^{\mathfrak{A}'_2}$ and therefore $[\![c_{a_1} = c_{a_2}]\!]^{\mathfrak{A}'_2} = 1$. Since $\mathfrak{A}'_2$ is a model of the diagram of $\mathfrak{A}_1$, $[\![c_{a_1} = c_{a_2}]\!]^{\mathfrak{A}_1^*} = 1$. Hence $[\![a_1 = a_2]\!]^{\mathfrak{A}_1} = 1$. Since $\mathfrak{A}_1$ is duplicate resistant, $a_1 = a_2$. Hence f is injective.
Finally, it is routine to check that f together with the identity function on B_1 witnesses that $\mathfrak{A}_1$ is isomorphic to $\mathfrak{A}_3$. □

5.3 Elementary Submodel and Downward Löwenheim-Skolem

Definition 5.7 (Elementary Submodel) Let $\mathfrak{A}_1$ be a B_1-valued model and $\mathfrak{A}_2$ be a B_2-valued model. $\mathfrak{A}_1$ is an *elementary submodel* of $\mathfrak{A}_2$ just in case: $\mathfrak{A}_1$ is a submodel of $\mathfrak{A}_2$, and for any formula $\phi(v_1, ..., v_n)$ of $\mathscr{L}$, any $a_1, ..., a_n \in A_1$,[16]

$$[\![\phi(v_1, ..., v_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n] = [\![\phi(v_1, ..., v_n)]\!]^{\mathfrak{A}_2}[a_1, ..., a_n]$$

[16] Here and in the next theorem we are ignoring the difference between a Boolean value and its image under an isomorphism between Boolean algebras.

Theorem 5.5 Let $\mathfrak{A}_1$ be a witnessing B_1-valued model and $\mathfrak{A}_2$ be a B_2-valued model. $\mathfrak{A}_1$is an elementary submodel $\mathfrak{A}_2$ if and only if $\mathfrak{A}_1$ is a submodel of $\mathfrak{A}_2$, and for any formula $\exists v\phi(v, v_1, ..., v_n)$ of $\mathscr{L}$, any $a_1, ..., a_n \in A_1$, for some $a \in A_1$,

$$[\![\exists v\phi(v, v_1, ..., v_n)]\!]^{\mathfrak{A}_2}[a_1, ..., a_n] = [\![\phi(v, v_1, ..., v_n)]\!]^{\mathfrak{A}_2}[a, a_1, ..., a_n]$$

Proof The left to right direction is proved by directly applying Definition 5.7 and the fact that $\mathfrak{A}_1$ is witnessing.The right to left direction is proved by induction on the complexity of ϕ. □

Definition 5.8 (Elementary Diagram) Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $\mathscr{L}_{\mathfrak{A}} = \mathscr{L} \cup \{c_a \mid a \in A\}$, where $\{c_a \mid a \in A\}$ is a new set of constants, one for each $a \in A$. Expand $\mathfrak{A}$ to a model of $\mathscr{L}_{\mathfrak{A}}$ (call it $\mathfrak{A}^*$) such that for all $a \in A$, $[\![c_a]\!]^{\mathfrak{A}^*} = a$.
The *elementary diagram* of $\mathfrak{A}$ is the B-valuation S^B which consists of and only of all the pairs of the form $\langle\phi, [\![\phi]\!]^{\mathfrak{A}^*}\rangle$ where ϕ is a sentence of $\mathscr{L}_{\mathfrak{A}}$ and $[\![\phi]\!]^{\mathfrak{A}^*}$ is the value of ϕ in $\mathfrak{A}^*$.

Theorem 5.6 Let $\mathfrak{A}_1$ be a B_1-valued model and $\mathfrak{A}_2$ be a B_2-valued model. The following statements are equivalent:

(1) $\mathfrak{A}_1$ is isomorphic to an elementary submodel of $\mathfrak{A}_2$.
(2) $\mathfrak{A}_2$ can be expanded to a model of the elementary diagram of $\mathfrak{A}_1$.

Proof The same proof as that of Theorem 5.4 with minor adjustments. □

When $\mathfrak{A}_1$ is isomorphic to an elementary submodel of $\mathfrak{A}_2$, we say that $\mathfrak{A}_1$ is *elementarily embedded* in $\mathfrak{A}_2$.
In Section 4 we proved a weaker version of the generalized Downward-Löwenheim-Skolem Theorem (Corollary 4.7.3). With the notion of elementary submodels we can now prove a stronger version of this theorem.

Theorem 5.7 (Strong-Downward-Löwenheim-Skolem) Let $\mathfrak{A}$ be an witnessing B-valued model of size $\geqslant |\mathscr{L}|$. Then $\mathfrak{A}$ has an elementary submodel of size $|\mathscr{L}|$.

Proof Let ϕ be an arbitrary sentence of $\mathscr{L}$ that is of the form $\exists v\psi$. Since $\mathfrak{A}$ is witnessing, there is some $a \in A$ such that $[\![\exists v\psi]\!]^{\mathfrak{A}} = [\![\psi]\!]^{\mathfrak{A}}[a]$. Pick such a witness for each sentence of the form $\exists v\psi$. Let $X \subseteq A$ be the set of all picked witnesses. Construct an increasing sequence:

$$X = X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq X_\alpha \subseteq ..., \ \ \alpha < |\mathscr{L}|$$

Let α be a successor ordinal. Let $\exists v\psi(v, v_1, ..., v_n)$ be a formula with $v_1, ..., v_n$ free, and let $a_1, ..., a_n \in X_{\alpha-1}$. Since $\mathfrak{A}$ is witnessing, there is some $a \in A$ such that $[\![\exists v\psi]\!]^{\mathfrak{A}}[a_1, ..., a_n] = [\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n]$. We pick a witness for each formula of the form $\exists v\psi(v, v_1, ..., v_n)$ and $a_1, ..., a_n \in X_{\alpha-1}$. Let X_α be $X_{\alpha-1}$ plus all the picked witnesses.
If α is a limit ordinal, let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.
Let $A' = \bigcup_{\alpha<|\mathscr{L}|} X_\alpha$. It is easy to check that each X_α has size $|\mathscr{L}|$. Hence $|A'| = |\mathscr{L}|$. Form a model $\mathfrak{A}'$ with universe A':

1. For any $a, b \in A'$, $[\![a = b]\!]^{\mathfrak{A}'} = [\![a = b]\!]^{\mathfrak{A}}$.
2. Let P be an n-ary relation. For any $a_1, ..., a_n \in A'$, $[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}'} = [\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}}$.
3. Let c be a constant. Let $[\![c]\!]^{\mathfrak{A}'}$ be some $a \in A'$ such that $[\![v_i = c]\!]^{\mathfrak{A}}[a] = [\![\exists v_i v_i = c]\!]^{\mathfrak{A}}$. Such an a exists by the setup of $\mathfrak{A}'$.

It is easy to see that $\mathfrak{A}'$ is a submodel of $\mathfrak{A}$. We show that $\mathfrak{A}'$ is also an elementary submodel of $\mathfrak{A}$. Let $\exists v\psi(v, v_1, ..., v_n)$ be a formula with $v_1, ..., v_n$ free, and let $a_1, ..., a_n \in A'$. Since $a_1, ..., a_n \in A' = \bigcup_{i<|\mathscr{L}|} X_i$, for some $i < |\mathscr{L}|$, $a_1, ..., a_n \in X_i$. Hence for some $a \in X_{i+1} \subseteq A'$, $[\![\exists v\psi]\!]^{\mathfrak{A}}[a_1, ..., a_n] = [\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n]$. By Theorem 5.5, $\mathfrak{A}'$ is an elementary submodel. □

The stronger Downward-Löwenheim-Skolem Theorem is a natural generalization of the homonymous theorem on two-valued models, as every two-valued model is witnessing. Interestingly, the requirement that $\mathfrak{A}$ is witnessing in the stronger Downward-Löwenheim-Skolem Theorem cannot be dropped, as the theorem no longer holds when $\mathfrak{A}$ is not necessarily witnessing. This result, I think, is another example of the fact that certain features of two-valued models can only be generalized to witnessing Boolean-valued models, but not to all Boolean-valued models.

Theorem 5.8 There exists a uncountable Boolean-valued model of a countable language that does not have a countable elementary submodel.

Proof Let B be a complete Boolean algebra such that from some $D \subseteq B$, $|D| = \omega_1$ and for any $C \subseteq D$ such that $|C| < \omega_1$, $\bigsqcup C \neq \bigsqcup D = p$. Let $D = \{p_\alpha \mid \alpha < \omega_1\}$. Let $|A| = \omega_1$. Let $A = \{a_\alpha \mid \alpha < \omega_1\}$. Let P be a unary predicate. (Predicates of other arities can work as well) Let $\mathfrak{A}$ be such that for any $\alpha < \omega_1$, $[\![P(a_\alpha)]\!]^{\mathfrak{A}} = p_\alpha$. The obviously $[\![\exists v P(v)]\!]^{\mathfrak{A}} = \bigsqcup_{\alpha<\omega_1} p_\alpha = \bigsqcup D = p$. And no countable submodel of $\mathfrak{A}$ is such that the value of $\exists v P(v)$ in it is p. □

5.4 Elementary Equivalence and Elementary Chain

Definition 5.9 (Elementary Equivalence) Let $\mathfrak{A}_1$ be a B_1-valued model and $\mathfrak{A}_2$ be a B_2-valued model. $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *elementarily equivalent* iff there is an isomorphism $h : B_1 \to B_2$ and for any sentence ϕ in $\mathscr{L}$, $[\![\phi]\!]^{\mathfrak{A}_1} = h([\![\phi]\!]^{\mathfrak{A}_2})$.

Theorem 5.9 Let I be an index set. For each $i \in I$, let $\mathfrak{A}_i$ be a witnessing B_i-valued model. Also, for any $i, j \in I$, let $\mathfrak{A}_i$ and $\mathfrak{A}_j$ be elementarily equivalent. Then there exists a model $\mathfrak{A}$ such that for any $i \in I$, $\mathfrak{A}_i$ is elementarily embedded in $\mathfrak{A}$.

Proof WLOG we assume all the $\mathfrak{A}_i$'s have the same value range B. For each $\mathfrak{A}_i$, let S_i^B be the elementary diagram of $\mathfrak{A}_i$. We assume that if $i \neq j$, then $\{c_a \mid a \in A_i\} \cap \{c_a \mid a \in A_j\} = \emptyset$. Let $\bigcup_{i\in I} S_i^B$ be the union of all the elementary diagrams.

Claim 5.9.1 $\bigcup\limits_{i\in I} S_i^B$ is a consistent B-valuation.

Proof of the Claim By Theorem 4.3, we only need to show that every finite sub-valuation of $\bigcup_{i\in I} S_i^B$ is consistent. Let $\Delta^B = \{\langle\phi_1(c_1), p_1\rangle, ..., \langle\phi_n(c_n), p_n\rangle\}$ be a finite sub-valuation of $\bigcup_{i\in I} S_i^B$. WLOG we assume that for any $1 \leqslant k \leqslant n$, $\langle\phi_k(c_k), p_k\rangle \in S_k^B$, and c_k is the only constant from $\{c_a \mid a \in A_k\}$ that appears in ϕ_k.

Assume for reductio that Δ^B is inconsistent. Then for some homomorphism $h : B \to 2$, Δ_h^B is inconsistent.

Suppose $\Delta_h^B = \{\theta_1(c_1), ..., \theta(c_n)\}$ such that $\theta_k = \phi_k$ if $h(p_k) = 1$ and $\theta_k = \neg\phi_k$ if $h(p_k) = 0$. Then $\theta_1(c_1) \vdash \neg\theta_2(c_2) \vee ... \vee \neg\theta_n(c_n)$.

Since $\langle\phi_1(c_1), p_1\rangle \in S_1^B$, $\theta(c_1) \in (S_1^B)_h$. Hence $(S_1^B)_h \vdash \neg\theta_2(c_2) \vee ... \vee \neg\theta_n(c_n)$. And by assumption $c_2, ..., c_n$ do not appear in $(S_1^B)_h$, hence $(S_1^B)_h \vdash \forall v_i \neg\theta_2(v_i) \vee ... \vee \forall v_i \neg\theta_n(v_i)$.

By assumption, $\forall v_i \neg\theta_2(v_i), ..., \forall v_i \neg\theta_n(v_i)$ are sentences of $\mathscr{L}$. Hence for each $2 \leqslant k \leqslant n$, for some $q_k \in B$, $\langle\forall v_i \neg\theta_k(v_i), q_k\rangle \in S_1^B$. Also since S_1^B is consistent (as it has a B-valued model, namely $\mathfrak{A}_1$), q_k is unique.

But all the $\mathfrak{A}_i$'s are elementarily equivalent. Hence for any $i \in I$, for any $2 \leqslant k \leqslant n$, $\langle\forall v_i \neg\theta_k(v_i), q_k\rangle \in S_i^B$. And as a result, for any $i \in I$, $\langle\forall v_i \neg\theta_2(v_i) \vee ... \vee \forall v_i \neg\theta_n(v_i), q_2 \sqcup ... \sqcup q_n\rangle \in S_i^B$.

Now since $(S_1^B)_h \vdash \forall v_i \neg\theta_2(v_i) \vee ... \vee \forall v_i \neg\theta_n(v_i)$, and since S_1^B is consistent, $h(q_2 \sqcup ... \sqcup q_n) = 1$. Hence for some $2 \leqslant k \leqslant n$, $h(q_k) = 1$.

Hence $\forall v_i \neg\theta_k(v_i) \in (S_k^B)_h$. But $\theta_k(c_k)$, by assumption, is also in $(S_k^B)_h$. Hence $(S_k^B)_h$ is inconsistent. But S_k^B is the elementary diagram of $\mathfrak{A}_\mathfrak{k}$, and therefore it has a B-valued model and should be consistent. Contradiction.

We showed that $\bigcup_{i\in I} S_i^B$ is consistent. By Theorem 4.6, it has a B-valued model $\mathfrak{A}'$. Let $\mathfrak{A}$ be the reduct of $\mathfrak{A}'$ to $\mathscr{L}$. By Theorem 5.6, for any $i \in I$, $\mathfrak{A}_i$ is elementarily embedded in $\mathfrak{A}$. $\square$

Definition 5.10 Let I be an index set. For each $i \in I$, let $\mathfrak{A}_i$ be a B_i-valued model. Then the direct product model, $\prod_{i\in I} \mathfrak{A}_i$, of the $\mathfrak{A}_i$'s, is defined as the following $\prod_{i\in I} B_i$-valued[17] model:

1. The universe is $\prod_{i\in I} A_i$, where for each i, A_i is the universe of $\mathfrak{A}_i$.
2. Let $\langle a_i\rangle_{i\in I}, \langle b_i\rangle_{i\in I} \in \prod_{i\in I} \mathfrak{A}_i$, $[\![\langle a_i\rangle_{i\in I} = \langle b_i\rangle_{i\in I}]\!]^{\prod_{i\in I}\mathfrak{A}_i} = \langle[\![a_i = b_i]\!]^{\mathfrak{A}_i}\rangle_{i\in I}$.
3. Let $\langle a_i^1\rangle_{i\in I}, \langle a_i^2\rangle_{i\in I}, ..., \langle a_i^n\rangle_{i\in I} \in \prod_{i\in I} \mathfrak{A}_i$, $[\![P(\langle a_i^1\rangle_{i\in I}, \langle a_i^2\rangle_{i\in I}, ..., \langle a_i^n\rangle_{i\in I})]\!]^{\prod_{i\in I}\mathfrak{A}_i} = \langle[\![P(a_i^1, a_i^2, ..., a_i^3)]\!]^{\mathfrak{A}_i}\rangle_{i\in I}$.
4. For any constant c in $\mathscr{L}$, $[\![c]\!]^{\prod_{i\in I}\mathfrak{A}_i} = \langle[\![c]\!]^{\mathfrak{A}_i}\rangle_{i\in I}$.

Theorem 5.10 (Direct Product Theorem) Let I be an index set. For each $i \in I$, let $\mathfrak{A}_i$ be a B_i-valued model. Let $\prod_{i\in I} \mathfrak{A}_i$ be their direct product model. Given an assignment $x : Var \to \prod_{i\in I} A_i$ on $\prod_{i\in I} \mathfrak{A}_i$, for each $i \in I$, let $y_i : Var \to A_i$ be the assignment on $\mathfrak{A}_i$ such that for any $v_n \in Var$, $y_i(v_n) = proj_i(x(v_n))$, where $proj_i : \prod_{i\in I} A_i \to A_i$ is the ith projection function. Then, for any formula ϕ in $\mathscr{L}$,

$$[\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}_i}[x] = \langle[\![\phi]\!]^{\mathfrak{A}_i}[y_i]\rangle_{i\in I}$$

[17] $\prod_{i\in I} B_i$ is the product algebra of the B_i's. It is easy to see that $\prod_{i\in I} B_i$ is a complete Boolean algebra when every B_i is a complete Boolean algebra.

Proof By induction on the complexity of ϕ. $\square$

Theorem 5.11 Let $\mathfrak{A}$ be a B-valued model. Let I be an arbitrary index set. Then $\mathfrak{A}$ is elementarily embedded in $\prod_{i\in I}\mathfrak{A}$.

Proof Let $B' = \{\langle p\rangle_{i\in I} \in \prod_{i\in I} B \mid p \in B\}$. It is easy to check that B' is isomorphic to B and B' is a complete subalgebra of $\prod_{i\in I} B$.
Let $\mathfrak{A}'$ be the B'-valued submodel of $\prod_{i\in I}\mathfrak{A}$ generated by $A' = \{\langle a\rangle_{i\in I} \mid a \in A\}$. It is easy to show that the domain of $\mathfrak{A}'$ is precisely A', since for any constant c, $[\![c]\!]^{\prod_{i\in I}\mathfrak{A}} = \langle [\![c]\!]^{\mathfrak{A}}\rangle_{i\in I} \in A'$. Also, for any formula $\phi(v_1, ..., v_n)$, any $\langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}$, $[\![\phi]\!]^{\mathfrak{A}'}[\langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}] = \langle [\![\phi]\!]^{\mathfrak{A}}[a_1, ..., a_n]\rangle_{i\in I} \in B'$, so the value range of $\mathfrak{A}'$ is indeed B'.
We can then show that $\mathfrak{A}'$ is an elementary submodel of $\prod_{i\in I}\mathfrak{A}$ by induction on the complexity of ϕ. The only non-trivial case is the inductive step on existential formulas, which holds by Theorem 5.10.
Let $f : A \to A'$ be such that for any $a \in A$, $f(a) = \langle a\rangle_{i\in I}$. It is easy to show that f witnesses that $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic. $\square$

Lemma 5.11.1 Let I be an index set. For any $i \in I$, let $\mathfrak{A}_i$ be a B_i-valued model that is witnessing. Then $\prod_{i\in I}\mathfrak{A}_i$ is a witnessing model.

Proof For simplicity we ignore the parameters. Let $\phi(v_i)$ be a formula. Then $[\![\exists v_i\phi]\!]^{\prod_{i\in I}\mathfrak{A}_i} = \langle [\![\exists v_i\phi]\!]^{\mathfrak{A}_i}\rangle_{i\in I}$, by Theorem 5.10. Since for any $i \in I$, $\mathfrak{A}_i$ is witnessing, for some $a_i \in A_i$, $[\![\exists v_i\phi]\!]^{\mathfrak{A}_i} = [\![\phi]\!]^{\mathfrak{A}_i}[a_i]$. Pick such an a_i for each $\mathfrak{A}_i$. Then $\langle [\![\exists v_i\phi]\!]^{\mathfrak{A}_i}\rangle_{i\in I} = \langle [\![\phi]\!]^{\mathfrak{A}_i}[a_i]\rangle_{i\in I} = [\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}_i}[\langle a_i\rangle_{i\in I}]$. $\square$

Theorem 5.12 Let $\mathfrak{A}$ be a witnessing B-valued model. Let I be an arbitrary index set. Let $h : \prod_{i\in I} B \to B$ be a homomorphism such that for any $p \in B$, $h(\langle p\rangle_{i\in I}) = p$. Then $\mathfrak{A}$ and $(\prod_{i\in I}\mathfrak{A})^h$ are elementarily equivalent.

Proof Let $\mathfrak{A}$ be a witnessing model and let $h : \prod_{i\in I} B \to B$ be a homomorphism such that for any $p \in B$, $h(\langle p\rangle_{i\in I}) = p$. Let ϕ be a sentence of $\mathscr{L}$. Let $[\![\phi]\!]^{\mathfrak{A}} = p \in B$.

By Lemma 5.11.1, $\prod_{i\in I}\mathfrak{A}$ is a witnessing model. Hence it is in the scope of Theorem 5.2. Hence $[\![\phi]\!]^{(\prod_{i\in I}\mathfrak{A})^h} = h([\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}}) = h(\langle [\![\phi]\!]^{\mathfrak{A}}\rangle_{i\in I}) = h(\langle p\rangle_{i\in I}) = p$. $\square$

Definition 5.11 (Chain of Models) Let α be an ordinal. For each $\beta < \alpha$, let $\mathfrak{A}_\beta$ be a B-valued model. A chain of models is an increasing sequence of models $\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset ... \subset \mathfrak{A}_\beta \subset ...$, $\beta < \alpha$, where $\mathfrak{A}_0$ is a submodel of $\mathfrak{A}_1$, $\mathfrak{A}_1$ is a submodel of $\mathfrak{A}_2$, etc.

Definition 5.12 (Union of the Chain) Given a chain of models $\mathfrak{A}_0 \subset ... \subset \mathfrak{A}_\beta \subset ...$, $\beta < \alpha$, the union of the chain is the B-valued model $\mathfrak{A} = \bigcup_{\beta<\alpha}\mathfrak{A}_\beta$ such that:

1. The universe of $\mathfrak{A}$ is $A = \bigcup_{\beta<\alpha} A_\beta$.
2. Let $a_1, a_2, ..., a_n \in A$. Then for some $\beta < \alpha$, $a_1, ..., a_n \in A_\beta$.
   (a) Let $1 \leqslant i, j, \leqslant n$. $[\![a_i = a_j]\!]^{\mathfrak{A}} = [\![a_i = a_j]\!]^{\mathfrak{A}_\beta}$.
   (b) Let P be an n-ary relation. $[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}} = [\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}_\beta}$.
   (c) Let c be a constant. $[\![c]\!]^{\mathfrak{A}} = [\![c]\!]^{\mathfrak{A}_\beta}$.

Proposition 5.1 The union of a chain is a B-valued model. Also, for every $\beta < \alpha$, $\mathfrak{A}_\beta$ is a submodel of $\bigcup_{\beta<\alpha} \mathfrak{A}_\beta$.

Theorem 5.13 (Generalized Elementary Chain Theorem) Let $\{\mathfrak{A}_\beta \mid \beta < \alpha\}$ be an elementary chain of models. Then for any $\beta < \alpha$, $\mathfrak{A}_\beta$ is an elementary submodel of $\bigcup_{\beta<\alpha} \mathfrak{A}_\beta$.

Proof Let $\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$. We need to show that for any $\beta < \alpha$, for any formula $\phi(v_1, ..., v_n)$, any $a_1, ..., a_n \in A_\beta$,

$$[\![\phi]\!]^{\mathfrak{A}}[a_1, ..., a_n] = [\![\phi]\!]^{\mathfrak{A}_\beta}[a_1, ..., a_n]$$

The atomic cases are already covered by Proposition 5.1. The inductive cases on sentential connectives are straightforward. Let $\phi(v_1, ..., v_n) = \exists v\psi(v, v_1, ..., v_n)$.
Let $[\![\exists v\psi]\!]^{\mathfrak{A}}[a_1, ..., a_n] = \bigsqcup_{a\in A}[\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] = p_1 \in B$. Let $[\![\exists v\psi]\!]^{\mathfrak{A}_\beta}[a_1, ..., a_n] = \bigsqcup_{a\in A}[\![\psi]\!]^{\mathfrak{A}_\beta}[a, a_1, ..., a_n] = p_2 \in B$.
Since $\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$, $A_\beta \subseteq A$. By inductive hypothesis we have $p_2 \leqslant p_1$. Hence we only need to show that $p_1 \leqslant p_2$.
Suppose $p_1 \nleqslant p_2$. Then for some $a \in A$, $[\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] \nleqslant p_2$. Let $[\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n]$ be p_3.
Since $a \in A = \bigcup_{\beta<\alpha} A_\beta$, for some $\eta < \alpha$, $a \in A_\eta$. Either $\eta \leqslant \beta$ or $\beta \leqslant \eta$. We will show that both possibilities lead to contradiction.
Suppose $\eta \leqslant \beta$. Then $a, a_1, ..., a_n \in A_\beta$. By inductive hypothesis, $[\![\psi]\!]^{\mathfrak{A}_\beta}[a, a_1, ..., a_n] = [\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] = p_3$. But then $p_3 \leqslant p_2 = [\![\exists v\psi]\!]^{\mathfrak{A}_\beta}[a_1, ..., a_n]$. Contradiction.
Suppose $\beta \leqslant \eta$. Then $a, a_1, ..., a_n \in A_\eta$. By inductive hypothesis, $[\![\psi]\!]^{\mathfrak{A}_\eta}[a, a_1, ..., a_n] = [\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] = p_3$. But since $a_1, ..., a_n \in A_\beta$, and $\mathfrak{A}_\beta$ is an elementary submodel of $\mathfrak{A}_\eta$ by the construction of the chain,

$$[\![\exists v\psi]\!]^{\mathfrak{A}_\eta}[a_1, ..., a_n] = [\![\exists v\psi]\!]^{\mathfrak{A}_\beta}[a_1, ..., a_n] = p_2$$

But then $p_3 \leqslant p_2$. Contradiction.
Hence $p_1 \leqslant p_2$. And therefore $p_1 = p_2$. □

6 True Identity Models

The identity symbol in Boolean-valued models is interpreted in a non-standard way. When B is a complete Boolean algebra that properly extends 2, our definition of Boolean-valued models allows that in some B-valued model $\mathfrak{A}$, for some $a, b \in A$, $[\![a = b]\!]^{\mathfrak{A}} = p \in B$, where p is neither 1_B or 0_B. This is an interesting feature of Boolean-valued models, which I believe will give rise to attractive philosophical applications. But that is a topic of another paper. In this section, nevertheless, we will the Boolean-valued models in which the identity symbol is interpreted in a standard way. Recall that in Section 3 we've defined the truth identity models:

Definition 6.1 A B-valued model $\mathfrak{A}$ is a *true identity model* just in case $[\![=]\!]^{\mathfrak{A}} : A \times A \to B$ is the real identity function on $A \times A$, i.e. for any $a, b \in A$, if a and b are not the same element, then $[\![a = b]\!]^{\mathfrak{A}} = 0_B$.

Proposition 6.1 Let $\mathscr{L}$ be a first order language whose only non-logical symbols are constants. Let $\mathfrak{A}$ be a B-valued true identity model of $\mathscr{L}$. Then for any formula $\phi(v_1, ..., v_n) \in \mathscr{L}$, any $a_1, ..., a_n \in A$, $[\![\phi]\!]^{\mathfrak{A}}[a_1, ..., a_n] \in \{0_B, 1_B\}$.

Theorem 6.1 Let $\mathfrak{A}$ be a B-valued true identity model. Let $h : B \to C$ be a homomorphism. Then the quotient model $\mathfrak{A}^h$ is a C-valued true identity model. Moreover, $\mathfrak{A}$ and $\mathfrak{A}^h$ have the same domain.

Proof $A = A^h$ because for any $a_1, a_2 \in A$, $a_1 \equiv_h a_2$ iff $h([\![a_1 = a_2]\!]^{\mathfrak{A}}) = 1$ iff $a_1 = a_2$, as $\mathfrak{A}$ is a true identity model. Also, if $[a_1]_h \neq [a_2]_h$, then $a_1 \neq a_2$, and then $[\![[a_1]_h = [a_2]_h]\!]^{\mathfrak{A}^h} = h([\![a_1 = a_2]\!]^{\mathfrak{A}}) = h(0_B) = 0_C$. □

We next define another special kind of Boolean-valued models - the full models.

Definition 6.2 (Antichain) Let B be a Boolean algebra. A subset $D \subseteq B$ is an *antichain* just in case for any $p \in D$, $p \neq 0$ and for any $p, q \in D$, $p \sqcap q = 0$.

Definition 6.3 (Full Model) Let $\mathfrak{A}$ be a B-valued model. $\mathfrak{A}$ is a *full* model just in case for any antichain $D \subseteq B$, and $\{a_d \mid d \in D\} \subseteq A$, there is an $a \in A$ such that for any $d \in D$, $d \leqslant [\![a = a_d]\!]^{\mathfrak{A}}$.

We can show that all full models are witnessing.

Theorem 6.2 Let $\mathfrak{A}$ be a full B-valued model. Then $\mathfrak{A}$ is witnessing.

Proof For simplicity we ignore the parameters. Let $\phi(v)$ be a formula with only v free. Let $[\![\exists v\phi]\!]^{\mathfrak{A}} = p \in B$. We will show that for some $a \in A$, $[\![\phi(v)]\!]^{\mathfrak{A}}[a] = p$. If $p = 0$, then the statement is trivial. So we assume $p > 0$.
Let $D = \{d \in B \setminus \{0\} \mid \text{for some } a^d \in A, d \leqslant [\![\phi(a^d)]\!]^{\mathfrak{A}}\}$. Let Q be the set of all antichains made up of elements in D. By Zorn's lemma, Q has a maximal element. Call it C.
We can show that D is dense below p. Let $0 \neq p' \leqslant p$. Since $p = \bigsqcup_{a \in A} [\![\phi(a)]\!]^{\mathfrak{A}}$, for some $a \in A$, $p' \sqcap [\![\phi(a)]\!]^{\mathfrak{A}} \neq 0$. But $p' \sqcap [\![\phi(a)]\!]^{\mathfrak{A}} \in D$ and $p' \sqcap [\![\phi(a)]\!]^{\mathfrak{A}} \leqslant p'$.
Hence $p \leqslant \bigsqcup C$: suppose not, then $p \sqcap -(\bigsqcup C) \neq 0$. Since D is dense below p, for some $d \in D$, $d \leqslant p \sqcap -(\bigsqcup C) \leqslant -(\bigsqcup C)$. Then $C \cup \{d\}$ is an antichain in D that properly extends C. Contradiction.
For every $d \in C$, let a^d be some element in A such that $d \leqslant [\![\phi(a^d)]\!]^{\mathfrak{A}}$.
Since $\mathfrak{A}$ is full, there is some $a \in A$ such that for all $d \in C$, $d \leqslant [\![a = a^d]\!]^{\mathfrak{A}}$.
Since $d \leqslant [\![\phi(a^d)]\!]^{\mathfrak{A}}$ as well, $d \leqslant [\![a = a^d]\!]^{\mathfrak{A}} \sqcap [\![\phi(a^d)]\!]^{\mathfrak{A}} \leqslant [\![\phi(a)]\!]^{\mathfrak{A}}$. Hence $p = [\![\exists v\phi]\!]^{\mathfrak{A}} \leqslant \bigsqcup C \leqslant [\![\phi(a)]\!]^{\mathfrak{A}}$. And trivially $[\![\phi(a)]\!]^{\mathfrak{A}} \leqslant [\![\exists v\phi]\!]^{\mathfrak{A}}$. Hence $[\![\phi(a)]\!]^{\mathfrak{A}} = [\![\exists v\phi]\!]^{\mathfrak{A}}$. □

But the next results show that the converse is false: witnessing models are not necessarily full.

Theorem 6.3 Let $\mathfrak{A}$ be a B-valued true identity model. If B is a proper Boolean extension of 2, and if $|A| > 1$, then $\mathfrak{A}$ is not a full model.

Proof Since B is a proper extension of 2, there is some $p \in B$ such that $0 \neq p \neq 1$. Then $\{p, -p\}$ is an antichain. Let a_1, a_2 be any two different elements in A. Then for any $a \in A$, either $p \nleqslant [\![a = a_1]\!]^{\mathfrak{A}}$, or $-p \nleqslant [\![a = a_2]\!]^{\mathfrak{A}}$, as $\mathfrak{A}$ is a true identity model. □

Theorem 6.4 Let $\mathscr{L}$ be an arbitrary first order language. Let B be a complete Boolean algebra that properly extends 2. Then there is a witnessing B-valued true identity model $\mathfrak{A}$ of $\mathscr{L}$, whose domain has more than one element.

Proof Pick $p \in B$ such that $0 \neq p \neq 1$. For any n-ary relation P in $\mathscr{L}$, for any $a_1, ..., a_n \in A$, let $[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}} = p$. Also let $[\![=]\!]^{\mathfrak{A}}$ be the identity function on $A \times A$. It is easy to show that $\mathfrak{A}$ is witnessing. □

Corollary 6.4.1 Let $\mathscr{L}$ be an arbitrary first order language. Let B be a complete Boolean algebra that properly extends 2. Then there is a witnessing B-valued true identity model of $\mathscr{L}$ that is not full.

In Section 4 we have reviewed a collection of results involving theories of first order languages and Boolean-valued models. In the following we will state a few theorems about theories and Boolean-valued true identity models. We will state the results without proofs as they are all very straightforward.

Theorem 6.5 Let T be a theory in $\mathscr{L}$. T is consistent if and only if for any complete Boolean Algebra B, T has a B-valued true idenity model $\mathfrak{A}$.

Theorem 6.6 Let B be any complete Boolean algebra. A theory T has a B-valued true identity model just in case every finite subset of T has a B-valued true identity model.

Recall that in Section 4, we have argued that the notion of Boolean-valuations is a natural generalization of the notion of theories. For the rest of this section we consider questions involving Boolean-valuations and true identity models. For example, what kind of Boolean-valuations correspond to true identity models? Does compactness holds on these Boolean-valuations?
To answer these questions we first need some familiar notions. For any $n < \omega$, let E_n be the first-order sentence that says there are at least n things,[18] and let $E!_n$ be the first-order sentence that says there are exactly n things.[19]

Definition 6.4 A B-valuation S^B of $\mathscr{L}$ *respects identity* iff S^B can be extended to a consistent B-valuation S'^B such that

1. For any constants $c, d \in \mathscr{L}$, either $\langle c = d, 1\rangle \in S'^B$ or $\langle c = d, 0\rangle \in S'^B$.
2. It is either the case that for some $n < \omega$, $\langle E!n, 1\rangle \in S'^B$, or the case that for every $n < \omega$, $\langle En, 1\rangle \in S'^B$.

[18] The sentence $\exists v_1, ..., v_n(\bigwedge_{1 \leqslant i < j \leqslant n} v_i \neq v_j)$ when $n > 1$, and the sentence $\exists v(v = v)$ when $n = 1$.

[19] That is, the sentence $E_n \wedge M_n$, where $M_n = \exists v_1, ..., v_n \forall u(u = v_1 \vee ... \vee u = v_n)$, for any $n < \omega$.

We now show that Boolean-valuations that respect identity are precisely those that have true identity models. To this end we need a series of lemmas.

Lemma 6.6.1 If a B-valuation S^B has a true identity B-valued model, then it respects identity.

Proof Suppose S^B has a true identity B-valued model $\mathfrak{A}$. Then the elementary diagram $S^B_{\mathfrak{A}}$ has either $\langle c = d, 1\rangle \in S'^B$ or $\langle c = d, 0\rangle$ for any $c, d \in \mathscr{L}$. Also, if $|A| = n$ for some $n < \omega$, then $\langle E!n, 1\rangle \in S^B_{\mathfrak{A}}$; if A is infinite, then for every $n < \omega$, $\langle En, 1\rangle \in S^B_{\mathfrak{A}}$. □

Lemma 6.6.2 Let $\mathfrak{A}_2$ be a B-valued model. Let $\mathfrak{A}_1$ be a B-valued submodel of $\mathfrak{A}_2$ such that for any $a \in A_2$,

$$\bigsqcup_{a_i \in A_1} [\![a = a_i]\!]^{\mathfrak{A}_2} = 1$$

Then $\mathfrak{A}_1$ is an elementary submodel of $\mathfrak{A}_2$.

Proof We can show that for any $\phi \in \mathscr{L}$, $[\![\phi]\!]^{\mathfrak{A}_1} = [\![\phi]\!]^{\mathfrak{A}_2}$ by induction on the complexity of ϕ. The only case worth mentioning is the quantifier case. For simplicity we omit the parameters.

$$\begin{aligned} [\![\exists v\phi]\!]^{\mathfrak{A}_2} &= \bigsqcup_{a\in A_2} [\![\phi(a)]\!]^{\mathfrak{A}_2} = \bigsqcup_{a\in A_2} \bigsqcup_{a_i\in A_1} [\![\phi(a)]\!]^{\mathfrak{A}_2} \sqcap [\![a = a_i]\!]^{\mathfrak{A}_2} \\ &= \bigsqcup_{a_i\in A_1} [\![\phi(a_i)]\!]^{\mathfrak{A}_2} = \bigsqcup_{a_i\in A_1} [\![\phi(a_i)]\!]^{\mathfrak{A}_1} = [\![\exists v\phi]\!]^{\mathfrak{A}_1} \end{aligned}$$

□

Lemma 6.6.3 Let S^B be a consistent B-valuation be such that (1) either $\langle c = d, 1\rangle \in S^B$ or $\langle c = d, 0\rangle \in S^B$, for any constants $c, d \in \mathscr{L}$, and (2) for some $n < \omega$, $\langle E!n, 1\rangle \in S^B$. Then S^B has a true identity B-valued model.

Proof Let $\langle E!n, 1\rangle \in S^B$. Let $C = \{c_1, ..., c_m\}$ be a set of constants in $\mathscr{L}$ such that for any $c_i, c_j \in C$, $\langle c_i = c_j, 0\rangle \in S^B$, and for any $d \in \mathscr{L}$, for some $c_i \in C$, $\langle c_i = d, 1\rangle \in S^B$. Since $\langle E!n, 1\rangle \in S^B$, $m \leqslant n$. If $m = n$, relabel the constants in C as $c_1, ..., c_n$. If $m < n$, first add new constants $c_{m+1}, ..., c_n$ to the language. Let $S'^B = S^B \cup \{\langle c_i = c_j, 0\rangle \mid 1 \leqslant i < j \leqslant n\}$. Since for any homomorphism $h : B \to 2$, $E!n \in (S'^B)_h$ (defined as in Definition 4.4), S'^B is consistent.
Let $\mathfrak{M}'$ be a B-valued model of S'^B. Let $\mathfrak{M}$ be its duplicate resistant copy. For any $1 \leqslant i \leqslant n$, let $[\![c_i]\!] = m_i \in M$. Let $\mathfrak{N}$ be the B-valued submodel of $\mathfrak{M}$ generated by $N = \{m_i \mid 1 \leqslant i \leqslant n\}$. It is straightforward to check that the domain of $\mathfrak{N}$ is N. Also obviously $\mathfrak{N}$ is a true identity model.
Since $\mathfrak{M} \models E!n$ and $\mathfrak{M} \models c_i \neq c_j$ for any $1 \leqslant i < j \leqslant m$, $\mathfrak{M} \models \forall v(v = c_1 \vee ... \vee v = c_n)$. Hence for any $m \in M$, $\bigsqcup_{1\leqslant i\leqslant n} [\![m = m_i]\!] = 1$. By Lemma 6.6.2, $\mathfrak{N}$ is an elementary submodel of $\mathfrak{M}$. Hence $\mathfrak{N}$ is a true identity B-valued model of S^B. □

Lemma 6.6.4 Let S^B be a consistent B-valuation be such that (1) either $\langle c = d, 1\rangle \in S^B$ or $\langle c = d, 0\rangle \in S^B$, for any constants $c, d \in \mathscr{L}$, and (2) for every $n < \omega$, $\langle En, 1\rangle \in S^B$. Then S^B has a true identity B-valued model.

Proof Let C be the set of constants in $\mathscr{L}$. Let D be a new set of constants with cardinality $|\mathscr{L}|$. Let $\mathscr{L}' = \mathscr{L} \cup D$. Enumerate all formulas with one free variable in $\mathscr{L}'$:

$$\phi_0, \phi_1, ..., \phi_\alpha, ... \qquad \alpha < |\mathscr{L}|$$

We now construct an increasing sequence of consistant B-valuations of $\mathscr{L}'$:

$$S^B = S_0 \subseteq S_1 \subseteq ... \subseteq S_\alpha \subseteq ... \qquad \alpha < |\mathscr{L}|$$

together with an increasing sequence of constants in $\mathscr{L}'$:

$$C = D_0 \subseteq D_1 \subseteq ... \subseteq D_\alpha \subseteq ... \qquad \alpha < |\mathscr{L}|$$

Successor stage. Suppose S_α and D_α are defined. Let $D_\alpha = \{c_i \mid i \in I\}$. We first find $p \in B$, $\{p_i \mid i \in I\} \subseteq B$ such that $S'_\alpha = S_\alpha \cup \{\langle \exists v\phi_\alpha, p\rangle\} \cup \{\langle \phi_\alpha(c_i), p_i\rangle \mid i \in I\}$ is consistent. The existence of these values is guaranteed by Theorem 4.5.
Let $d_\alpha \in D$ be a new constant that hasn't appeared in S'_α. Let $S^+_\alpha = S'_\alpha \cup \{\langle c_i = d_\alpha, 0\rangle \mid c_i \in D_\alpha\}$. Since for every $n < \omega$, $\langle En, 1\rangle \in S'_\alpha$, every finite subset of S^+_α is consistent. Hence S^+_α is consistent.
Now let X be the set of all homomorphisms h from B to 2 and let $K = \{\Delta \mid \Delta \text{ is a finite subset of } S^+_\alpha\}$.
Similar to what we do in the proof of Theorem 4.4:

1. For any $h \in X$, any $\Delta = \{\langle \phi_1, p_1\rangle, ..., \langle \phi_k, p_k\rangle\} \in K$, let $q^h_\Delta = q_1 \sqcap ... \sqcap q_k$, where for any $1 \leqslant i \leqslant k$, $q_i = p_i$ if $h(p_i) = 1$, and $q_i = -p_i$ if $h(p_i) = 0$.
2. Let $J^+_\Delta = \{h' \in X \mid \Delta_{h'} \vdash \phi_\alpha(d_\alpha)\}$ and $J^-_\Delta = \{h^* \in X \mid \Delta_{h^*} \vdash \neg\phi_\alpha(d_\alpha)\}$.
3. Let $q^+_\Delta = \bigsqcup_{h' \in J^+_\Delta} q^{h'}_\Delta$ and $q^-_\Delta = \bigsqcup_{h^* \in J^-_\Delta} q^{h^*}_\Delta$.

Claim 6.6.1

$$\bigsqcup_{\Delta \in K} q^-_\Delta \;\sqcap\; \bigsqcap_{i \in I} -p_i \;\sqcap p = 0$$

Proof of the Claim It suffices to show that for any $\Delta \in K$, for any $h^* \in J^-_\Delta$,

$$q^{h^*}_\Delta \;\sqcap\; \bigsqcap_{i \in I} -p_i \;\sqcap p = 0$$

Suppose not. Then for some $h \in X$, $h(p) = 1$, $h(q^{h^*}_\Delta) = 1$ and $h(p_i) = 0$ for every $i \in I$. Hence $\exists v\phi_\alpha \in (S^+_\alpha)_h$, and for every $c_i \in D_\alpha$, $\neg\phi_\alpha(c_i) \in (S^+_\alpha)_h$.
Since $h(q^{h^*}_\Delta) = 1$, by the reasoning as in the proof of Claim 4.4.2, $\Delta_{h^*} = \Delta_h$. Hence $\Delta_h \vdash \neg\phi_\alpha(d_\alpha)$. Hence $(S^+_\alpha)_h \vdash \neg\phi_\alpha(d_\alpha)$.

Hence $(S'_\alpha)_h$ and some finite subset of $\{c_i \neq d_\alpha \mid c_i \in D_\alpha\}$ together entail $\neg\phi_\alpha(d_\alpha)$. Since d_α does not appear in $(S'_\alpha)_h$, for some finite set $\{c_1, ..., c_m\} \subseteq D_\alpha$,

$$(S'_\alpha)_h \vdash \forall v(v \neq c_1 \wedge ... \wedge v \neq c_m \rightarrow \neg\phi_\alpha(v))$$

But since $\exists v\phi_\alpha \in (S^+_\alpha)_h$ and for every $c_i \in D_\alpha$, $\neg\phi_\alpha(c_i) \in (S^+_\alpha)_h$, $(S^+_\alpha)_h$ is inconsistent, contradicting that S^+_α is consistent.

By Claim 4.4.2, $\bigsqcup_{\Delta \in K} q^+_\Delta \sqcap \bigsqcup_{\Delta \in K} q^+_\Delta = 0$.
Let $r = \bigsqcup_{\Delta \in K} q^+_\Delta \sqcup (\bigsqcap_{i \in I} -p_i \sqcap p)$. Hence $\bigsqcup_{\Delta \in K} q^-_\Delta \sqcap r = 0$.
Let $S_{\alpha+1} = S^+_\alpha \cup \{\langle \phi_\alpha(d_\alpha), r\rangle$. By the same reasoning as in the proof of Theorem 4.4, the above observation shows that $S_{\alpha+1}$ is consistent. Moreover,

$$\begin{aligned}\bigsqcup_{i \in I} p_i \sqcup r &\geqslant \bigsqcup_{i \in I} p_i \sqcup \left(\bigsqcap_{i \in I} -p_i \sqcap p\right)\\ &= \bigsqcup_{i \in I} p_i \sqcup p = p\end{aligned}$$

as $p_i \leqslant p$ for any $i \in I$ in order for S'_α to be consistent.
Hence $\bigsqcup_{c_i \in D_\alpha} [\![\phi_\alpha(c_i)]\!]^{S_{\alpha+1}} \sqcup [\![\phi_\alpha(d_\alpha)]\!]^{S_{\alpha+1}} = [\![\exists v\phi_\alpha]\!]^{S_{\alpha+1}}$. In other words, we guarantee that in $S^{\alpha+1}$ there are instances of ϕ_α that “collectively witness" (the value of) $\exists_v\phi_\alpha$.
Finally Let $D_{\alpha+1} = D_\alpha \cup \{d_\alpha\}$.
For the limit stage, we take the union: i.e. we let $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. Since every S_β is consistent, S_α is consistent by Theorem 4.3.
Let $T' = \bigcup_{\alpha<|\mathscr{L}|} S_\alpha$. Again by Theorem 4.3, T' is consistent. Extend T' to a maximal consistent B-valuation T^B (Theorem 4.5).
We construct a B-valued model $\mathfrak{M}$ as follows:

1. The universe $M = C \cup D$.
2. For any constant $c \in \mathscr{L}'$, let $[\![c]\!]^{\mathfrak{M}} = c$.
3. For any n-nary relation P, any $d_1, ..., d_n \in M$, $[\![P(d_1, ..., d_n)]\!]^{\mathfrak{M}} = [\![P(d_1, ..., d_n)]\!]^{T^B}$.
4. For any $c, d \in M$, $[\![c = d]\!]^{\mathfrak{A}} = [\![c = d]\!]^{T^B}$.

Claim 6.6.2 $\mathfrak{M}$ is a model of T^B.

Proof of the Claim By an argument similar to that in the proof of Theorem 4.6. The only difference worth mentioning is the inductive case of quantified formulas. Let $[\![\exists v\theta]\!]^{T^B} = p$. At some stage α in the construction, $\langle \exists v\theta, p\rangle$ is added, together with some instance-value pairs. By the set up of the stages,

$$[\![\exists v\theta]\!]^{T^B} = \bigsqcup_{d \in D_\alpha} [\![\theta(d)]\!]^{T^B}$$

By inductive hypothesis, for each $d \in D_\alpha$, $[\![\theta(d)]\!]^{\mathfrak{M}} = [\![\theta(d)]\!]^{T^B}$. Hence $[\![\exists v\theta]\!]^{\mathfrak{M}} = [\![\exists v\theta]\!]^{T^B}$.

Now let $\mathscr{L}^- = \mathscr{L} \cup \bigcup_{\alpha<|\mathscr{L}|} D_\alpha$. Let $\mathfrak{M}^-$ be the reduct of $\mathfrak{M}$ to $\mathscr{L}^-$. Hence $\mathfrak{M}^-$ is a model of S^B. Let $\mathfrak{N}$ be the submodel of $\mathfrak{M}^-$ generated by $N = \bigcup_{\alpha<|\mathscr{L}|} D_\alpha$. Clearly the domain of $\mathfrak{N}$ is N.

Also $\mathfrak{N}$ is a true identity model, since for any $d_\alpha \in \bigcup_{\alpha<|\mathscr{L}|} D_\alpha$, at the stage when it is added, for any constant $c \in \mathscr{L}^-$ that already exists in the previous stages, $\langle c = d_\alpha, 0\rangle$ is also added.

Finally, let $d \in M \setminus N$, i.e. $d \in D \setminus \bigcup_{\alpha<|\mathscr{L}|} D_\alpha$. Then at some stage S_α in the construction, the pair $\langle \exists v(v = d), 1\rangle$ is added. Hence $\bigsqcup_{c\in D_\alpha} [\![c = d]\!]^{S_\alpha} = 1$. Hence $\bigsqcup_{c\in N} [\![c = d]\!]^{\mathfrak{M}} = 1$. By Lemma 6.6.2, then, $\mathfrak{N}$ is an elementary submodel of $\mathfrak{M}$. Hence $\mathfrak{N}$ is a true identity B-valued model of S^B. □

Theorem 6.7 A B-valuation S^B respects identity if and only if it has a true identity B-valued model.

Proof By Lemmas 6.6.1, 6.6.3 and 6.6.4. □

Theorem 6.8 A B-valuation S^B respects identity if and only if every finite sub-valuation of S^B respects identity.

Proof The left to right direction is obvious.

For the right to left direction, let S^B be such that every finite subset of S^B respects identity. By Theorem 6.7, every finite subset of of S^B has a true identity model.

We first show that for any $c, d \in \mathscr{L}$, either $S^B \cup \langle c = d, 1\rangle$ or $S^B \cup \langle c = d, 0\rangle$ is such that every finite subset of it has a true identity model. Suppose neither. Then for some finite subsets $\Delta_1, \Delta_2 \subseteq S^B$, both $\Delta_1 \cup \langle c = d, 1\rangle$ and $\Delta_2 \cup \langle c = d, 0\rangle$ have no true identity model. Hence $\Delta_1 \cup \Delta_2$ has no true identity model. Contradiction.

Hence we can extend S^B to a consistent B-valuation S^+ such that for every $c, d \in \mathscr{L}$, either $\langle c = d, 1\rangle$ or $\langle c = d, 0\rangle \in S^+$.

Claim 6.8.1 It is either the case that (a) for some $n < \omega$, every finite subset Δ of S^+ is consistent with $\langle E!n, 1\rangle$, or the case that (b) for every $n < \omega$, every finite subset Δ of S^+ is consistent with $\langle En, 1\rangle$.

Proof of the Claim Suppose neither case holds. Then for every $n < \omega$, some finite subset Δ_n of S^+ is inconsistent with $\langle E!n, 1\rangle$, and for some $m < \omega$, some finite subset Δ' of S^+ is inconsistent with $\langle Em, 1\rangle$.

Let $\Delta = \Delta_1 \cup ... \cup \Delta_{m-1} \cup \Delta'$. Then Δ is inconsistent with $\{\langle E!1, 1\rangle, ..., \langle E!m - 1, 1\rangle, \langle Em, 1\rangle\}$. Then for some $h : B \to 2$, Δ_h is inconsistent with $\{E!1, ..., E!m - 1, Em\}$, which means that Δ_h is inconsistent, contradicting that S^+ is consistent.

If (a), then let $(S')^B = S^+ \cup \{\langle E!n, 1\rangle\}$. If (b), then let $(S')^B = S^+ \cup \{\langle Em, 1\rangle \mid m < \omega\}$. Hence S^B respects identity. □

Corollary 6.8.1 A B-valuation S^B has a true identity B-valued model if and only if every finite sub-valuation of S^B has a true identity B-valued model.

7 Löwenheim-Skolem Theorems

In previous sections we have proved two versions of the downward Löwenheim-Skolem Theorem:

Theorem 7.1 If a Boolean-valuation S^B of $\mathscr{L}$ has a B-valued model, then it has a witnessing B-valued model of size $\leqslant |\mathscr{L}|$.

Theorem 7.2 Let $\mathfrak{A}$ be an witnessing B-valued model of size $\geqslant |\mathscr{L}|$. Then $\mathfrak{A}$ has an elementary submodel of size $|\mathscr{L}|$.

A natural question is: what about the upward Löwenheim-Skolem Theorem? Can it also be generalized to a Boolean-valued setting? In this section we investigate this question.
The case of the upward Löwenheim-Skolem is much more complicated than its downward counterpart. Recall that in Section 5 we observed that our definition of Boolean-valued models allow there to be "null" duplicates in a model. And with the existence of null duplicates it is boringly easy to add more objects to a domain of a model without changing which sentences are true in the model:

Theorem 7.3 Let T be a consistent theory of $\mathscr{L}$. Then for any complete Boolean algebra B, if T has a B-valued model of size α, it has B-valued models of arbitrary sizes larger than α.

Proof Just pick some random element of the domain and add as many duplicates of the element to the domain as we want. □

Note that the above theorem is much stronger than the normal upward Löwenheim-Skolem in the two-valued cases. It says that any consistent theory can have models that are arbitrarily large, including, for example, a theory that says there are only two objects. This is a counter-intuitive result. Surely if a sentence saying that there are only two objects is true in a model, then we would want there to be only two objects in the domain of the model.
One might think that the culprit of this counter-intuitive result is the existence of duplicates. What if we require the models to be duplicate resistant (Definition 5.1)? Will it still be the case that a theory that demands finite bivalent models can have arbitrarily large Boolean-valued models? The answer, interestingly, is positive, as the following results show.
We first observe that one sentence that does have control over the size of a duplicate resistant Boolean-valued model is the sentence saying that there is at most one thing.

Theorem 7.4 Let $\mathfrak{A}$ be a duplicate resistant model. If $\mathfrak{A} \models \exists v_1 \forall v_2 (v_1 = v_2)$, then $|A| = 1$.

Proof $[\![\exists v_1 \forall v_2 (v_1 = v_2)]\!]^{\mathfrak{A}} = \bigsqcup_{a \in A} \bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}}$. Fix some $a \in A$. Consider $\bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}}$. We will show that $\bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}} = \bigsqcap_{c,d \in A} [\![c = d]\!]^{\mathfrak{A}}$. The $\geqslant$ direction holds trivially. The $\leqslant$ direction holds as for any $c, d \in A$, $[\![a = c]\!]^{\mathfrak{A}} \sqcap [\![a = d]\!]^{\mathfrak{A}} \leqslant [\![c = d]\!]^{\mathfrak{A}}$. Hence $\bigsqcup_{a \in A} \bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}} = \bigsqcup_{a \in A} \bigsqcap_{c,d \in A} [\![c = d]\!]^{\mathfrak{A}} = \bigsqcap_{c,d \in A} [\![c = d]\!]^{\mathfrak{A}} = 1$. Hence for any $c, d \in A$, $[\![c = d]\!]^{\mathfrak{A}} = 1$. Since $\mathfrak{A}$ is duplicate resistant, c and d are the same element. □

Nevertheless, that there is at most one thing is also the *only* sentence that has control over the size of a duplicate resistant Boolean-valued model:

Definition 7.1 Let $D \subseteq B$ be an antichain. Then D is a maximal antichain iff $\bigsqcup D = 1$.

Lemma 7.4.1 Let $\mathfrak{M}$ be a B-valued duplicate resistant model. Let B' be a complete Boolean algebra such that B is a complete subalgebra of B'. Let $D \subseteq B$ be a maximal antichain s.t. for any $a \neq b \in M$, and $d \in D$, $d \sqcap [\![a \neq b]\!]^{\mathfrak{M}} \neq 0$. Then $\mathfrak{M}$ is elementarily embedded in a duplicate resistant B'-valued model of size $|M|^{|D|}$.

Proof Let $N = \{f \mid f : D \to M\}$. We construct a B'-valued model with domain N as follows:

1. For any $f_1, f_2 \in N$, let $[\![f_1 = f_2]\!]^{\mathfrak{N}} = \bigsqcup_{d \in D}(d \sqcap [\![f_1(d) = f_2(d)]\!]^{\mathfrak{M}})$.
2. For any n-ary relation symbol P, any $f_1, ..., f_n \in N$, $[\![P(f_1, ..., f_n)]\!]^{\mathfrak{N}} = \bigsqcup_{d \in D}(d \sqcap [\![P(f_1(d), ..., f_n(d))]\!]^{\mathfrak{M}})$.
3. For any constant c, $[\![c]\!]^{\mathfrak{N}} = g_m$, where m is the referent of c in $\mathfrak{M}$, and $g_m(d) = m$ for any $d \in D$.

Claim 7.4.1 $\mathfrak{N}$ is a duplicate resistant B'-valued model.

Proof of the Claim Let $f_1, f_2, f_3 \in N$. For any $d \in D$, $d \sqcap [\![f_1(d) = f_2(d)]\!]^{\mathfrak{M}} \sqcap [\![f_2(d) = f_3(d)]\!]^{\mathfrak{M}} \leqslant d \sqcap [\![f_1(d) = f_3(d)]\!]^{\mathfrak{M}}$. Hence $[\![f_1 = f_2]\!]^{\mathfrak{N}} \sqcap [\![f_2 = f_3]\!]^{\mathfrak{N}} \leqslant [\![f_1 = f_3]\!]^{\mathfrak{N}}$. The clause on relation symbols is similar. The other clauses on identity are straightforward. Hence $\mathfrak{N}$ is a B'-valued model.
If $f_1 \neq f_2 \in N$, then for some $d \in D$, $f_1(d) \neq f_2(d)$. By the assumption on D, then, $d \sqcap [\![f_1(d) \neq f_2(d)]\!]^{\mathfrak{M}} \neq 0$. But $(d \sqcap [\![f_1(d) \neq f_2(d)]\!]^{\mathfrak{M}}) \sqcap [\![f_1 = f_2]\!]^{\mathfrak{N}} = 0$. Hence $[\![f_1 = f_2]\!]^{\mathfrak{N}} \neq 1$. Hence $\mathfrak{N}$ is a duplicate resistant model.

Now we show that $\mathfrak{M}$ is elementarily embedded in $\mathfrak{N}$. The embedding $i : M \to N$ is such that for any $m \in M$, $i(m) = g_m$, where $g_m(d) = m$ for any $d \in D$.
Atomic formulas. It is easy to check that $[\![g_{m_1} = g_{m_2}]\!]^{\mathfrak{N}} = \bigsqcup_{d \in D} d \sqcap [\![m_1 = m_2]\!]^{\mathfrak{M}} = [\![m_1 = m_2]\!]^{\mathfrak{M}}$, as D is maximal. The case for relations is similar.
The inductive case for connectives is routine. For the quantifier case, first we observe that for any $f \in N$,

$$\bigsqcup_{m \in M} [\![f = g_m]\!]^{\mathfrak{N}} = 1$$

This is because for any $d \in D$, it is easy to check that $d \leqslant [\![f = g_{f(d)}]\!]^{\mathfrak{N}}$. Since D is an maximal antichain, $\bigsqcup_{d \in D} [\![f = g_{f(d)}]\!]^{\mathfrak{N}} = 1$.

Hence (parameters are omitted for simplicity) $[\![\exists v \phi]\!]^{\mathfrak{N}} = \bigsqcup_{m \in M} [\![\phi(g_m)]\!]^{\mathfrak{N}}$, by an argument similar to that of Lemma 6.6.2. By inductive hypothesis, then, $[\![\exists v \phi]\!]^{\mathfrak{N}} = [\![\exists v \phi]\!]^{\mathfrak{M}}$. ◻

Theorem 7.5 Let T be a theory that is consistent with $\neg \exists v_1 \forall v_2 (v_1 = v_2)$. Then T has arbitrarily large duplicate resistant Boolean-valued models. In particular, if B is infinite, then T has an infinite duplicate resistant B-valued model of size $\geqslant \kappa$, where κ is the maximum size of maximal antichains in B.

Proof T has a classical bivalent model $\mathfrak{M}$. Let B be a complete Boolean algebra that property extends 2. Clearly 2 is a complete subalgebra of B. We first observe that since $\mathfrak{M}$ is classical, for any maximal antichain $D \subseteq B$, $a \neq b \in M$, any $d \in D$, $d \sqcap [\![a \neq b]\!]^{\mathfrak{M}} = d \sqcap 1 = d \neq 0$.
For any cardinal $\kappa > |M|$, there is a complete Boolean algebra B that has a maximal antichain D of size κ. Hence, by Lemma 7.4.1 and the above observation, T has a duplicate resistant B-valued model of size $|M|^{\kappa}$. Also, if B is infinite, then B has an infinite maximal antichain. □

The above result shows that duplicate resistant models also cannot truthfully describe the size of its domain. Let $E!n$ be the sentence that says there are exactly n things, where $n > 1$. As long as B properly extends 2, then there will be a B-valued duplicate resistant model of $E!n$ whose domain is larger than n. Indeed, the above result shows that there can be arbitrarily large duplicate resistant B-valued models of $E!n$.
Note that if we fix a complete Boolean algebra B, it is not always the case that any T that is consistent with $\neg\exists v_1 \forall v_2 (v_1 = v_2)$ has arbitrarily large duplicate resistant B*-valued* models. For example, let $B = \{0, p, -p, 1\}$ be the Boolean algebra with four elements. Let $\mathfrak{A}$ be a B-valued model of $E!2$. Then it is easy to check that for some $a \neq b \in A$, $[\![a = b]\!] = 0$.[20] Suppose $\mathfrak{A}$ is duplicate resistant and $c \neq a, b \in A$. Then either $[\![a = c]\!] = p$ and $[\![b = c]\!] = -p$, or $[\![a = c]\!] = -p$ and $[\![b = c]\!] = p$. Also, for any $c, d \in A$, if $[\![a = c]\!] = [\![a = d]\!]$, then $[\![c = d]\!] = 1$.[21] Hence $\mathfrak{A}$ has at most four elements.
We observe that duplicate resistant models in general do not truthfully describe the size of their domains. This is essentially because the identity symbol is still interpreted in a non-standard way. Hence, in the context of Boolean-value models, in order to get larger models that are really larger, we need to focus on true identity models, as these are the models in which identity is standard. Once we introduce this requirement, then, we can generalize the upward Löwenheim-Skolem theorem in the way that is most non-trivial. First it is easy to see that true identity models are the only ones that truthfully describe the size of their domains.

Observation 7.5.1 Let $\mathfrak{A} \models E!n$, where $n < \omega$. Then $|A| = n$ iff $\mathfrak{A}$ is a true identity model.

With the help of results from Section 6, we can generalize the upward Löwenheim-Skolem Theorems to true identity models:

Theorem 7.6 If a B-valuation S^B has an infinite B-valued true identity model $\mathfrak{A}$, then it has infinite B-valued true identity models of any power $\alpha \geqslant |A|$.

Proof Let $c_\beta, \beta < \alpha$ be a list of new constant. Consider the B-valuation $S'^B = S^B \cup \{\langle c_\gamma = c_\beta, 0\rangle \mid \gamma < \beta < \alpha\}$. Every finite subset of S'^B has a true identity B-valued model (namely $\mathfrak{A}$). By Corollary 6.8.1, S'^B has a B-valued true identity model, whose size has to be at least α. □

[20] Since $[\![E!2]\!] = 1$, $\bigsqcap_{a,b \in A} [\![a = b]\!] = 0$. Suppose $[\![a = b]\!] = p$ and $[\![a' = b']\!] = -p$. Consider $q = [\![a = a']\!]$. If $q = 0$, we are done. If $q = p$, then $[\![a = b']\!] = 0$. Similarly if $q = -p$, then $[\![a' = b]\!] = 0$. If $q = 1$, then $[\![a = b']\!] = -p$ and hence $[\![b = b']\!] = 0$. These are the only options.

[21] WLOG suppose $[\![a = c]\!] = [\![a = d]\!] = p$. Then $p \leqslant [\![c = d]\!]$. But $[\![b = c]\!] = [\![b = d]\!] = -p$. Hence $-p \leqslant [\![c = d]\!]$. Hence $[\![c = d]\!] = 1$.

Theorem 7.7 If a B-valuation S^B has arbitrarily large finite B-valued true identity models, then it has an infinite B-valued true identity model.

Proof The same proof as that of Theorem 7.6. □

Corollary 7.7.1 Every infinite true identity model has arbitrarily large elementary extensions.

As a special case of Theorems 7.6 and 7.7, we also have:

Theorem 7.8 If a theory T has arbitrarily large finite B-valued true identity models, then it has an infinite B-valued true identity model.

Theorem 7.9 If a theory T has an infinite B-valued true identity model $\mathfrak{A}$, then it has infinite B-valued true identity models of any power $\alpha \geqslant |A|$.

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