Boolean Mereology

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Abstract

Most ordinary objects - cats, humans, mountains, ships, tables, etc. - have indeterminate mereological boundaries. If the theory of mereology is meant to include ordinary objects at all, we need it to have some space for mereological indeterminacy. In this paper, we present a novel degree-theoretic semantics - Boolean semantics - and argue that it is the best degree-theoretic semantics for modeling mereological indeterminacy, for three main reasons: (a) it allows for incomparable degrees of parthood, (b) it enforces classical logic, and (c) it is compatible with all the axioms of classical mereology. Using Boolean semantics, we will also investigate the connection between vagueness in parthood and vagueness in existence/identity. We show that, contrary to what many have argued, the connection takes neither the form of entailment nor the form of exclusion.

1 Introduction

When we look around and inspect the ordinary objects around us, we will find that many ordinary objects lack a precise mereological boundary, or at least they appear to do so. Many ordinary objects are such that in certain natural situations, we can find things that are neither definitely part of it nor definitely not part of it. Here are some typical examples:

Example One Consider Tibbles the cat. Suppose Tibbles has a whisker, call it W, that has loosened up and is about to fall off. Is W part of Tibbles?

Example Two Consider Mount Kilimanjaro, the tallest mountain in Africa. Suppose there is a tree, call it T, that is located somewhere at the boundary of Kilimanjaro - say, somewhere in between Mweka Camp and Materuni Waterfall. Is T part of Kilimanjaro?

Example Three Consider Tim, an ordinary human being. Suppose there is a cell, call it C, in Tim's epidermis that has lost its nucleus and is about to be shed from the surface of Tim's skin. Is *C* part of Tim?

Example Four Consider Theseus the ship. Suppose there is an iron nail, call it N, that is in the process of being hammered into Theseus by a repairer. Is N part of Theseus?

There are countless other examples of this type, involving ordinary objects of almost all kinds, including animals, humans, artifacts, geographical areas, plants, buildings, and so on. If we describe the cases and ask the common man questions of the form "is W/T/C/N part of Tibbles/Kilimanjaro/Tim/Theseus?", the answer we would most likely get would be a hesitant "sort of/more or less/to some extent". These answers, I believe, are natural and intuitive. They indicate that an all-encompassing theory of the relation of parthood should have the ability to accommodate indeterminacy.

In this paper I will present a novel degree-theoretic semantic framework that is able to handle mereological indeterminacy with ease. The semantic framework I am about to introduce is called *Boolean-valued semantics*, whose key feature is that degrees of truths form a Boolean ordering. I will argue that Boolean-valued semantics is the best *degree-theoretic* semantics for the language of mereology. In particular, I will argue that it trumps the well-known alternative - fuzzy-valued semantics, for three main reasons: (a) it allows for incomparable degrees of parthood, (b) it enforces classical logic, and (c) it is compatible with all the axioms of classical mereology. Moreover, I will explore, under the framework of Boolean semantics, the connection between vagueness in parthood and vagueness in existence/identity. I will show that, contrary to many have argued, vagueness in parthood *entails* neither vagueness in existence nor vagueness in identity, although being compatible with both.

What I won't do in this paper, nevertheless, is to develop a full-fledged philosophical theory of mereological vagueness that has a decisive answer to every relevant question. The main goal of this paper is to construct a superior semantic framework for indeterminacy of parthood, and I believe that it should never be the job of the semantics to take a stand on deeper philosophical questions like "What is the nature of mereological indeterminacy?". An ideal semantic framework should be flexible to which philosophical viewpoints one further upholds. In the final section of this paper, I will illustrate the neutrality and flexibility of Boolean semantics by sketching out two different philosophical theories of mereological vagueness, one coming from applying Boolean semantics to the view that mereological vagueness is linguistic, and the other coming from applying Boolean semantics to the view that mereological vagueness is ontic. Another issue that I won't discuss in this paper is higher-order vagueness. In this chapter, I will adopt (without arguing) a McGee-style position that the issue of higher-order vagueness lies in the interpretation of the *meta-language*.[1] And since the the purpose of this chapter is to build a semantics, that is, an interpretation framework of the *object* language - the language of mereology, the issue of higher-order vagueness, on our assumption, lies outside of the scope of our discussion.

The plan of this paper goes as follows. I will start in Section Two by arguing that facing mereological vagueness, a natural, and good place to start is to adopt a degree-theoretic semantics. In Section Three, I will present in details Boolean semantics, which is a degree-theoretic semantics whose key feature is that truth degrees form a Boolean structure. I will explain how Boolean semantics can be applied to cases of mereological indeterminacy. In Section Four, I will argue that Boolean semantics is the better degree-theoretic semantics for handling mereological indeterminacy, in comparison to the alternative. The goal of Section Five is to investigate a special kind of Boolean models for the language of mereology that are of particular interest - the atomic Boolean models. Via these models I will also discuss the connection between mereological vagueness on the one hand and vagueness in existence and identity on the other hand. Finally, in section Six, we end this paper with a discussion on the nature of mereological vagueness. In particular, we show that Boolean mereology is neutral to what is the nature of mereological vagueness, and one can construct different theories of mereological vagueness by combining Boolean semantics with different views on the nature of mereological vagueness.

2 Many Degrees: A Natural Start

The language of mereology, depending on one's taste, is a first-order or second-order language whose only non-logical symbol is the binary relation symbol of parthood, $\precsim$. The classical semantics, for either first-order or second-order logic, has as its value range the two-valued Boolean algebra $\{0,1\}$. The classical semantics, therefore, leaves little if not no room for mereological indeterminacy, as, for example, W is either part of Tibbles to the degree 0, meaning that it is not part of Tibbles, or it is part of Tibbles to the degree 1, meaning that it is part of Tibbles. In order to accommodate mereological indeterminacy, therefore, we at least need revision of some kind to the classical semantics [2].

[1] See [16] and [18] in favor of arguments for this viewpoint and replies to objects.

[2] Although most people think that at least some change to classical semantics is needed for handling mereological indeterminacy, there are also exceptions. For epistemicists like Williamson [28], sentences like "W is part of Tibbles" do indeed have a definite truth value, and it is just impossible for us humans to know the truth values of these sentences. Mereological vagueness is explained, on this view, as a kind of ignorance that we cannot possibly overcome. Most people find this view highly counter-intuitive. Under this view, there will have to be basic mereological facts about ordi-

A natural and straightforward move is to enlarge the range of truth degrees. If "yes" corresponds to the degree 1 and "no" corresponds to the degree 0, then we might want some intermediate degree between 0 and 1 to correspond to the common man's hesitant "sort of", when responding to the question "is W part of Tibbles". If we have decided to add more degree of parthood, then, there seems to be no harm but only benefits if we add more than just one. Consider the case of Tibbles. It is certainly possible that there is a different whisker, call it W', that has also loosened up and is about to fall off. But we can imagine that W' is looser than W, and also has a stronger inclination to fall off. In this case, then, it seems quite intuitive to say that the extent to which W' is part of Tibbles is lower than the extent to which W is part of Tibbles. If we want to transform these "extent talks" to "degrees talks", we will then want to have multiple intermediate degrees that are comparable to each other, so that we can assign a lower intermediate degree to "W' is part of Tibbles" and a higher one to "W is part of Tibbles".

Let us call a semantic framework "degree-theoretic" if it allows for multiple degrees of truth in addition to the extreme ones. The semantic framework that I am about to develop, Boolean semantics, is a degree-theoretic one. There are, I believe, a number of advantages to use a degree-theoretic semantics on cases of mereological indeterminacy. First, under a degree-theoretic framework, the changes that need to be made to the classical semantics are quite unsubstantial and procedural. All we need to do is to replace the classical value range $\{0, 1\}$ with a value range of a larger size. The core idea behind the classical semantics story stays unchanged, including, for example, that constants in the language are interpreted by objects in the domain, that truth values are assigned to the atomic formulas by an assignment function that comes with the model, that complex formulas have their values calculated from the values of simpler formulas using certain algebraic operations, and so on[3]. What we end up with is a natural generalization of the classical semantics theory, rather than a radical deviation. Second, a degree-theoretic semantics offers at least some level of explanation of what mereological indeterminacy is. Under a degree-theoretic framework, cases of mereological indeterminacy are cases of intermediate parthood degrees, that is, cases where some object is part of another to an intermediate degree between 0 and 1. The phenomenon of indeterminacy is explained in terms of non-extreme truth degrees. Of course, this does not answer all the questions we care about mereological indeterminacy, such as, for example,

nary objects in the world that are simply epistemically inaccessible to us, no matter how our cognitive abilities improve. It seems to me to be a heavy philosophical burden to postulate these unreachable facts about the mereological relations among ordinary objects.

[3] Admittedly it is of course theoretically possible for there to be degree-theoretic views of mereological indeterminacy that are not truth functional. But to my knowledge in the current context this is not something worth of special discussion.

"What is the nature of mereological indeterminacy?", or "Is mereological indeterminacy worldly or not?". But it is a decent first step. Last but not least, as we have already observed, our ordinary intuition about the relation of parthood involves that it is susceptible to comparison. Among the two loosened up whiskers the looser one is less a part of Tibbles than the tighter one. Among the two trees at the boundary the further one is less a part of Kilimanjaro than the closer one. So on and so forth. Such intuitions can be neatly captured by a degree-theoretic semantics as long as we have multiple comparable intermediate degrees.

The above discussion is not meant to be a decisive argument against using non-degree-theoretic semantics for cases mereological indeterminacy. There is a variety of different non-degree-theoretic semantics, and I do not believe there is a sufficiently strong objection against them all. Each one has its own problems, and I will postpone the discussion of some of them to the later sections[4]. The above discussion is only meant to point out some general advantages enjoyed by having a degree-theoretic semantics, and that the latter is a good place to start, if our goal is to develop a semantics for the relation of parthood that tolerates indeterminacy.

3 What Are Boolean Degrees?

The classical value range $\{0,1\}$ is the two-element complete Boolean algebra, and in classical semantics, logical terms like "and", "or", etc. are interpreted by the algebraic operations - meet, join, etc. - on the Boolean algebra. If our plan is to enlarge the classical value range whereas keep the rest of classical semantics unchanged, then the natural suggestion is to use larger complete Boolean algebras as value range and still interpret logical terms using Boolean operations. Degrees of truth, then, form a complete Boolean algebra that has more than two elements.

Definition 3.1. A Boolean algebra[5] is a set B together with binary operations $\sqcap$ and $\sqcup$, unary operation $-$, and elements 0 and 1 that satisfies:

1. commutative and associative laws for $\sqcap$ and $\sqcup$;
2. distributive laws for $\sqcap$ over $\sqcup$ and $\sqcup$ over $\sqcap$;
3. for any $x, y \in B$, $x \sqcup (x \sqcap y) = x$; $x \sqcap (x \sqcup y) = x$; $x \sqcup -x = 1$; $x \sqcap -x = 0$.

In each Boolean algebra we can define an ordering $\leqslant$ as follows: for any $x, y \in B$, $x \leqslant y$ just in case $x \sqcap y = x$. We can show that this ordering is a partial order:

[4]For example, we will talk about supervaluation semantics and its connection to Boolean semantics in Section Six.

[5]For a detailed introduction to Boolean algebras, see [**?**].

in fact, it gives rise to a bounded distributive complemented lattice. 1 is the top element with respect to this ordering, and 0 is the bottom element with respect to this ordering[6].

Definition 3.2. A complete Boolean algebra B is a Boolean algebra where each subset of B has a supremum with respect to the ordering $\leqslant$.

In classical semantics, models are $\{0,1\}$-valued. In Boolean semantics, models are B-valued[7], where B can be any complete Boolean algebra. Just as in the classical case, a Boolean model $\mathfrak{A}$ comes with a pre-given set of objects, A, as its domain. Any constant in the language is interpreted by an object in the domain. The identity symbol is interpreted by a function from A^2 to B that satisfies the following conditions: for any $a_1, a_2, a_3 \in A$[8],

$$\llbracket a_1 = a_1 \rrbracket^{\mathfrak{A}} = 1$$
$$\llbracket a_1 = a_2 \rrbracket^{\mathfrak{A}} = \llbracket a_2 = a_1 \rrbracket^{\mathfrak{A}}$$
$$\llbracket a_1 = a_2 \rrbracket^{\mathfrak{A}} \sqcap \llbracket a_2 = a_3 \rrbracket^{\mathfrak{A}} \leqslant \llbracket a_1 = a_3 \rrbracket^{\mathfrak{A}}$$

An n-ary relation symbol P is interpreted by a function from A^n to B that satisfies the following conditions: for any $a_1, ..., a_n, b_1, ..., b_n \in A$[9],

$$\llbracket P(a_1,...,a_n) \rrbracket^{\mathfrak{A}} \sqcap \big(\bigsqcap_{1 \leqslant i \leqslant n} \llbracket a_i = b_i \rrbracket^{\mathfrak{A}} \big) \leqslant \llbracket P(b_1,...,b_n) \rrbracket^{\mathfrak{A}}$$

Again, just as in the classical case, the sentential connectives and quantifiers are interpreted by algebraic operations on the Boolean algebra: conjunction by binary meet, disjunction by binary join, negation by complementation, universal quantifier by infinite meet and existential quantifier by infinite join. In particular, given an assignment function x from the set of all variables to A, and suppose ϕ, ψ

[6]In fact, an alternative characterization of a Boolean algebra is a bounded distributive complemented lattice.

[7]For a more formal definition of a Boolean-valued model, see Def. A.1 in the Appendix.

[8]Here and in the following, for any sentence ϕ and any Boolean model $\mathfrak{A}$, $\llbracket \phi \rrbracket^{\mathfrak{A}}$ means the value of ϕ in $\mathfrak{A}$. We might omit the superscript occasionally when the context is clear.

[9]In any complete Boolean algebra B, for any $D \subseteq B$, $\bigsqcap D$ is the infimum of D with respect to the ordering $\leqslant$, whose existence is guaranteed by the definition of a complete Boolean algebra (with an easy derivation). Similarly, $\bigsqcup D$ is the supremum of D with respect to $\leqslant$.

are formulas,

$$
\begin{aligned}
[\![\neg\phi]\!]^{\mathfrak{A}}[x] &= -[\![\phi]\!]^{\mathfrak{A}}[x] \\
[\![\phi \wedge \psi]\!]^{\mathfrak{A}}[x] &= [\![\phi]\!]^{\mathfrak{A}}[x] \sqcap [\![\psi]\!]^{\mathfrak{A}}[x] \\
[\![\phi \vee \psi]\!]^{\mathfrak{A}}[x] &= [\![\phi]\!]^{\mathfrak{A}}[x] \sqcup [\![\psi]\!]^{\mathfrak{A}}[x] \\
[\![\exists v_i \phi]\!]^{\mathfrak{A}}[x] &= \bigsqcup_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x] &= \bigsqcap_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)]
\end{aligned}
$$

where $x(v_i/a)$ is the assignment function that takes v_i to a and agrees with x at everywhere else.

Now we have shown that Boolean semantics arises from classical semantics simply by replacing the two-element complete Boolean algebra of classical truth degrees with an arbitrary non-trivial complete Boolean algebra. After this change, we faithfully follow the classical procedure, step-by-step. The new value range can be as large as we want[10], as there can be arbitrarily large complete Boolean algebras. Therefore, there can be multiple intermediate degrees in between the top degree 1 and the bottom degree 0. Ordered by $\leqslant$, some of the intermediate degrees are higher/lower than some others. These Boolean degrees are perfect for modeling mereological indeterminacy. The whisker W_1 that is firmly attached to Tibbles is part of Tibbles to the degree 1; the whisker W_2 that has already fallen off from Tibbles is part of Tibbles to the degree 0; the whisker W that has loosened up and is inclined to fall off is part of Tibbles to the degree p, where p is an intermediate degree between 0 and 1 in a complete Boolean algebra that is sufficiently large; the whisker W' that is just like W except that it is looser and has a greater inclination to fall off is part of Tibbles to the degree q, where q is some intermediate degree between 0 and 1 that is strictly less than p. Boolean mereology centers around the simple idea that parthood comes in Boolean degrees. The basic thought behind the view is that while the classical picture does great in modeling the parthood relations among abstract mathematical objects like geometrical spheres or spacial-temporal regions that are perfectly precise, it is inadequate when we wish to further theorize about the parthood relations among ordinary objects like cats and mountains that have vague mereological boundaries. To deal with the ordinary objects we need a wider range of parthood degrees in addition to 0 and 1, and that wider range should be a larger complete Boolean algebra under Boolean mereology.

[10] This means that there is *no limitation* to how large a Boolean value range can be (i.e. there is no largest cardinal κ such that there is no complete Boolean algebra of size larger than κ). This does not mean, however, that for any cardinal κ, there is a complete Boolean algebra of size κ. For example, there is no complete Boolean algebra that has exactly five elements.

4 Why Boolean Degrees?

In the literature on mereological indeterminacy, or the literature on vagueness in general, the most mainstream, or even perhaps the only currently available version of degree-theoretic theory, is the one which changes the classical semantics by substituting the classical value range with the real interval $[0,1]$, ordered in the standard way. Let us call a degree-theoretic semantics of this kind, or just a degree-theoretic semantics under which the degrees of parthood are ordered linearly, a *fuzzy* semantics. Of course, my definition here of a fuzzy semantics is very general, and as it stands a cluster of views that differ from each other in bigger or smaller details satisfy this definition. But the points that I am about to make in the rest of this section should be applicable to them all.

Since any complete Boolean algebra larger than $\{0,1\}$ is not a linear order, Boolean semantics, in the sense that matters, is not a fuzzy semantics. Boolean semantics actually shares a lot in common with a fuzzy semantics. They both originate from the simple thought that the classical semantics is inadequate at modeling the mereological status of ordinary objects because it offers too few options. Therefore, they both plan to change the classical semantics by enlarging the value range and keep the rest untouched. The key difference, of course, is which structure we should replace the classical value range with. It is interesting to note that the classical value range $\{0,1\}$ is the only non-degenerate ordering that is both linear and Boolean. So both Boolean semantics and fuzzy semantics agree in that we should generalize some algebraic property of the classical value range in order to build larger ranges, but they disagree on which algebraic property we should generalize: for the fuzzy semantics, it is the property of being linear; for Boolean semantics, it is the property of being Boolean.

Despite sharing commonalities, Boolean mereology and the fuzzy ones differ in substantial ways. In the rest of this section, I will argue that Boolean semantics is the better degree-theoretic semantic framework when it comes to theorizing about mereological indeterminacy. Starting with a humble point, the biggest motivation behind the fuzzy views is that our intuition that parthood among ordinary objects is not an all-or-nothing matter; rather, it seems to come in different degrees. Common sense confirms that the tighter whisker W is part of Tibbles to an extent greater than that of the looser whisker W', though neither of the two whiskers are definitely part of Tibbles, as they are both on the verge of falling off. The biggest selling point of the fuzzy views, I think, is that it is able to capture this intuition. Under a fuzzy view, we can, for example, say that W is part of Tibbles to the degree 0.5 while W' is part of Tibbles to the degree 0.4; or in general, the tighter a shaky whisker is, the higher the degree we assign to it being part of Tibbles. But we can do the same thing with a Boolean ordering of truth degrees. Complete Boolean algebras can be as large as we want, and therefore there can be as many intermediate parthood degrees as want. As long as the Boolean value range has more than four elements, there will be two intermediate degrees p,q between 0 and 1 such that q is strictly less than p, so that we can let p be the degree to which W is part of Tibbles and q be the degree to which W' is part of Tibbles.

Second, although sometimes we have borderline cases of parthood whose degrees of parthood seem comparable, sometimes we have borderline cases of parthood whose degrees of parthood seem *incomparable*. Consider, for example, the tree T that is boundary of Mount Kilimanjaro. It is indeterminate whether T is part of Kilimanjaro, meaning that the degree to which T is part of Kilimanjaro is an intermediate value between 0 and 1, just as the degree to which the whisker W is part of Tibbles. But should the former degree be higher than the latter, or should the latter be higher than the former, or should they be equivalent? How exactly should we compare the degree to which T is part of Kilimanjaro to the degree to which W is part of Tibbles? I think it is impossible to answer these questions. Unlike in the case of W and W', there is simply *no* sensible dimension by which we can compare the degree to which T is part of Kilimanjaro and the degree to which W is part of Tibbles. The two degrees should be simply *incomparable*. It is absurd to assert that T is more part of Kilimanjaro than W is part of Tibbles and equally absurd to assert the opposite. But under a fuzzy semantics we have no choice but to have the two degrees be comparable to each other, since a linear ordering of degrees is connected, meaning that for any two fuzzy degrees p,q, either $p \leqslant q$ or $q \leqslant p$. This is, I believe, a unfortunate consequence of using a fuzzy semantics on mereology. And we can avoid it by adopting a Boolean semantics instead. Any complete Boolean algebra that is larger than $\{0,1\}$ is not connected, and therefore there will be elements p,q such that neither $p \leqslant q$ nor $q \leqslant p$. Boolean mereology thus has the resources to refrain from comparing the degree to which T is part of Kilimanjaro and the degree to which W is part of Tibbles. In short, under Boolean mereology, unlike under its fuzzy counterpart, we do not have to make incomparable comparisons.

Third, the most commonly held and perhaps the most powerful objection to the fuzzy views is that they are in tension with classical rules of reasoning[11]. Departing from classical logic, I believe, comes with great costs, for at least two reasons. First, classical rules and tautologies that are invalid under the fuzzy views - say, for example, the law of excluded middle - are widely endorsed and employed in almost all other areas in philosophy and in mathematics. Rejecting classical logic would mean that fuzzy mereology has to be an isolated, lonely bubble in the theory space. Second, the way in which the fuzzy views violate classical logic brings

[11] See, for exmaple, [13].

upon unwelcome consequences. For example, consider the sentence that W is part of Tibbles. The truth degree of this sentence has to be an intermediate value, since W is a borderline case. But by the reasoning, the negation of this sentence - that W is not part of Tibbles - also has to have an intermediate truth value. And because the values are ordered linearly, the conjunction of the two sentences - that W is both part of and not part of Tibbles - has to have an intermediate truth value as well, at least under the standard form of the fuzzy view. But that sounds wrong: nothing can be both part of and not part of Tibbles. The conjunction has the form of a contradiction, and a contradiction should be outright false instead of being somewhere in between truth and falsity.

Boolean mereology, in contrary, avoids all these problems, as it not only is compatible with but also enforces classical logic. As we will prove in the Appendix, Boolean-valued models, for first-order languages, for example, are sound and complete with respect to first-order logic. This means that all the theorems of first-order logic are true to the degree 1 in every Boolean-valued model.[12] Therefore, sentences like that W is both part of and not part of Tibbles always have degree 0 in Boolean-valued models. Similarly, sentence like that W is either part of Tibbles or not part of Tibbles always have degree 1. With Boolean truth degrees, we can have a many-degree truth-functional semantics with classical rules of inferences satisfied.

Last but not least, under Boolean mereology, not only can we have theorems of classical logic satisfied, we can also have principles of classical mereology satisfied. This point will be exemplified in the next section where we discuss a special kind of Boolean models for the language of mereology - the atomic Boolean models. Basically, we can have Boolean-valued models of mereology where all the principles of classical mereology have value 1. In contrast, this is something that is incredibly difficult, if not utterly impossible, to achieve, under the fuzzy approach.

For example, consider the case of Tibbles, of which W is a vague part. Under fuzzy semantics, the sentence that W is part of Tibbles should be a real number in $(0,1)$. Let's say that W is part of Tibbles to the degree 0.5. Now, clearly Tibbles is distinct from the whole mereological universe (whose existence is guaranteed by classical mereology): lots of things, the Eiffel Tower, for example, are part of

[12] In fact, it is proven in the Appendix that Boolean-valued models preserves classical *validity*, defined as preservation of having truth value 1 (Theorem B.3.1). An alternative way of defining validity in this context is to say that a sentence is a (Boolean) consequence of a set of premises just in case in every Boolean-valued model, the value of the sentence is larger than or equal to the supremum of the values of the premises. McGee and McLaughlin (in [18]) proved that in the context of propositional calculus, this alternative definition of Boolean consequence also coincides with that of logical consequence. My conjecture is that this equivalence also holds in the context of predicate calculus, since I see no difficulty of extending their proof to the first-order case.

Tibbles to the degree 0. A consequence of classical mereology - the principle of strong complementation[13] - says that everything that is distinct from the universe has a (mereological) complement. Since Tibbles is distinct from the universe to the degree 1, there has to be an object, call it Complement, such that it is the complement of Tibbles to the degree 1. This means that (1) Complement overlaps with Tibbles to the degree 0, and (2) the fusion of Tibbles and Complement is identical to the entire universe to the degree 1. But, then, what should be the degree to which W is part of Complement? In order for the degree to which Complement overlaps with Tibbles to be 0, the degree to which W is part of Tibbles and is part of Complement has to be 0, which means that the degree to which W is part of Complement can only be 0. But then the fusion of Tibbles and Complement is such that W is part of it to the degree 0.5, whereas the universe is such that W is part of it to the degree 1. So the fusion of Tibbles and Complement is not identical to the entire universe to the degree 1. Contradiction.

For similar but slightly more difficult reasons, we can see that even the principle of weak supplementation[14] is going to fail under fuzzy semantics. And it is not hard to see that the failure of these classical mereological principles under fuzzy semantics is essentially due to the linear ordering of the truth values. In the case of Tibbles and Complement, in order for the principle of strong complementation to be true, we need the degree x to which W is part of Complement to be such that the supremum of x and 0.5 is 1 and the infimum of x and 0.5 is 0. Nevertheless, when the truth values are linearly ordered, there simply is no such value. When the truth values form a Boolean ordering, on the other hand, such a value does exist, as we will see shortly below. In any case, the general point here is simply that by adopting a fuzzy semantics we will have to sacrifice part of classical mereology, and this is a sacrifice that cannot be ignored, as classical mereology is well-understood and deeply intertwined with other areas in contemporary metaphysics. We can avoid this sacrifice by adopting Boolean semantics instead.

5 Atomic Boolean Models

The goal of this section is to investigate a special kind of Boolean models for mereology, which we will call the atomic Boolean models. These models arise from a simple and natural idea. We start with a pre-given set of mereological atoms

[13]Formally, the principle of strong complementation is the following sentence in $\mathscr{L}_M$: $\forall v_1(\neg U(v_1) \to \exists v_2(\neg v_1 \circ v_2 \wedge \forall v_3(Fu(v_3, \{v_1, v_2\}) \to U(v_3)))))$, where $U(v_1) := \forall v_2(v_2 \precsim v_1)$. For the definition of $\mathscr{L}_M$ and other defined notions, see Def. 5.1.

[14]Formally, the principle of weak supplementation is the following sentence in $\mathscr{L}_M$: $\forall v_1 \forall v_2(v_1 \precnsim v_2 \to \exists v_3(v_3 \precsim v_2 \wedge \neg v_1 \circ v_3))$. For the definition of $\mathscr{L}_M$ and other defined notions, see Def. 5.1.

S. Then, taking a complete Boolean algebra B as value range, we let domains of the models consist of functions from S to B. Intuitively, any function $f : S \to B$ corresponds to an object composed of the mereological atoms. For any $a \in S$, $f(a)$ is the degree to which the atom a is part of (the object represented by) f.

The atomic Boolean models[15] are particularly interesting and worth studying for multiple reasons. First, as mentioned above, atomic Boolean models are intuitively motivated. If the world is built up from mereological atoms, and if mereological relations comes in degrees, then the natural picture is that every object in the world is composed of the atoms to certain degrees. That is, it should be the case that every object in the world can be represented by a function from the set of all atoms to Boolean degrees, which is exactly what atomic Boolean models are like. Second, as argued above, Boolean mereology, unlike the fuzzy views, is easily compatible with axioms of classical mereology. Below we will exemplify this point by showing that a special case of the atomic Boolean models - the $SEVI$ models - are models of the system CM, which is equivalent to classical mereology. So with Boolean semantics we can have a degree-theoretic semantics of mereology with all axioms of classical mereology satisfied.

Third, in the literature on vague mereology, there has been a fair amount of discussion on the relationship between vague parthood on the one hand, and vague existence and vague identity on the other hand[16]. Many, for example, have either argued or tacitly assumed that vague parthood entails vague existence, and therefore proponents of mereological vagueness are also stuck with existential vagueness. A study of atomic Boolean models, as I will show below, will shed light on how, under Boolean semantics, vague parthood is connected with vague existence and vague identity. In particular, I will show that their connection neither takes the form of entailment nor takes the form of exclusion, as there can be atomic Boolean models, though being models of vagueness, that disallow vagueness in existence/identity, and atomic Boolean models that allow vagueness in existence/identity.

Last but not least, I believe that atomic Boolean models are mathematically

[15]The atomic Boolean models, as we will see in a moment, are models of the axiom of Atomicity. This does not mean, however, that Boolean semantics are stuck with atomic mereology. Note that atomic Boolean models are a *special kind* of Boolean-valued models for mereology that naturally arise *on the assumption that* the world is atomic. There can certainly be other types of Boolean-valued models for mereology that model, for example, some kind of gunky mereology. We focus on atomic Boolean models here simply because of their simplicity and their effectiveness in illustrating our points, as will be listed below.

[16]Here's a non-comprehensive list of articles that have touched on these equestions: Evans [8], Weatherson [27], Barnes and Williams [2] have argued that vague parthood entails vague identity; Cook [5], Sainsbury [23], and some others have argued for the opposite; van Inwagen [26], Lewis [14], Smith [24], Merricks [19] and many others hold that vague parthood entails vague existence; Morreau [20] and Donnelly [6] hold the opposite view.

interesting to study as well. This is because atomic Boolean models are similar in multiple aspects to the standard Boolean-valued models of set theory, as presented in, say, Bell [3]. For example, the definition of the values of the atomic clauses on parthood in the atomic Boolean models is similar to the definition of the values of the clauses on subsethood in the Boolean models for set theory: the former is defined in terms of the degree to which every atom that is part of the first object is part of the second object, while the latter is defined in terms of the degree to which every element that is a member of the first set is a member of the second set. Another example is that when proving the holding of the axiom of Fusion in the atomic Boolean models, we construct a fusion in the same way as we construct a mixture of a collection of Boolean-valued sets. These commonalities in techniques perhaps hint towards a deeper connection between Boolean-valued parthood and Boolean-valued membership, which seems to be worth of further study.

We will divide the rest of this section into two subsections. We will devote the first subsection to presenting a version of the formal theory of mereology that is tailored specifically to our needs. In the second subsection, we will define properly different kinds of the atomic Boolean models, use them to explore the relationship between vagueness in parthood and vagueness in existence/identity, and discuss which axioms of classical mereology hold in these different kinds of atomic Boolean models.

5.1 Classical Mereology

As mentioned above, one of the primary goals of studying atomic Boolean models is to investigate the relation between vague parthood and vague existence/identity. We will also investigate how, given the presence of vague parthood, different axioms of classical mereology are connected with the presence/non-presence of vague existence/identity. But to meet these needs we will have to deviate from the standard formulation of classical mereology to some extent, for reasons I will explain in a moment. In particular, the deviation will come in two parts: (a) we will alter, in minor but important details, the way in which some non-primitive mereological notions are defined in terms of the notion of parthood, and (b) we will present and group the axioms of classical mereology in a way that is slightly more complicated and cumbersome than the standard.

Part (a) of the deviation further consists of two changes. The first, and the most important change we will make is that we will define an “existence” predicate and restrict quantification to objects that satisfy this predicate at certain places (for example, when defining “overlap”, “fusion”, etc.). The reason why we need this change is because the standard formulation of classical mereology tacitly assumes that everything in the domain of quantification fully exists, and therefore leaves no room for vague existence at all. In order to be able to discuss the *possibility* of vague existence, therefore, we have to define this "existence" predicate that serves the purpose of measuring the degree to which an object exists, and have it impact the domain of quantification at places that matter. The second change we will make is less non-trivial and is mostly just for convenience: we will define the notion of proper part without using the identity symbol. Later we will see that this small change allows all the axioms of atomic classical mereology except Anti-Symmetry to be formulated without the identity symbol. Therefore, it will follow directly from the formulation of these axioms that the truth/falsity of these axioms in a Boolean model is not affected by how identity is defined in the model, or in other words, whether we have vague identity or not.

Now we introduce the language of mereology and the defined notions:

Definition 5.1. The language of mereology, $\mathscr{L}_M$, is the second order language[17] whose signature contains a single binary relation $\lesssim$ (parthood). We further define the following relations in this language:

1. $v_1 \lnsim v_2 := v_1 \lesssim v_2 \wedge \neg v_2 \lesssim v_1$.
2. $E(v_1) := \exists v_2(\neg v_1 \lesssim v_2)$.
3. $v_1 \circ v_2 := \exists v_3(E(v_3) \wedge v_3 \lesssim v_1 \wedge v_3 \lesssim v_2)$.
4. $At(v_1) := E(v_1) \wedge \forall v_2(E(v_2) \rightarrow \neg v_2 \lnsim v_1)$.
5. $FU(v_1, X_1) = \forall v_2(X_1(v_2) \rightarrow v_2 \lesssim v_1) \wedge \forall v_3(v_3 \lesssim v_1 \wedge E(v_3) \rightarrow \exists v_4(X_1(v_4) \wedge v_3 \circ v_4))$.

Intuitively, $v_1 \lesssim v_2$ means that v_1 is a part of v_2. $v_1 \lnsim v_2$ means that v_1 is a proper part of v_2. $E(v_1)$ means that v_1 exists, or that v_1 is not zero, in the sense that v_1 is not a part of everything. $v_1 \circ v_2$ means that v_1 and v_2 overlap. $At(v_1)$ means that v_1 is a mereological atom. $FU(v_1, X_1)$ means that v_1 fuses the X_1's, i.e. that everything in X_1 is part of v_1. and everything that exists and is part of v_1 overlaps with something in X_1.

We now move on to axioms of mereology, which are sentences in $\mathscr{L}_M$. We divide these axioms into four groups, for purposes we will explain in a moment:

[17] Whether classical mereology should be formulated as a first-order or second-order theory is not a trivial issue, and one might have different preferences based on their other theoretical commitments. For example, a nominalist might want to avoid quantifying over second-order entities. But none of these concerns, I think, matter to our discussion of mereological indeterminacy. In this paper I define the theory of classical mereology as a second-order theory simply because this is the more demanding option, and all the Boolean constructions we have laid out in this paper can be easily carried over to the first-order case.

Definition 5.2. The *minimal* theory of Classical Mereology (*MCM*) contains the following three axioms:

$$\text{(Transitivity)} \qquad \forall v_1 \forall v_2 \forall v_3 (v_1 \precsim v_2 \wedge v_2 \precsim v_3 \rightarrow v_1 \precsim v_3)$$

$$\text{(Supplementation)} \qquad \forall v_1 \forall v_2 (v_2 \not\precsim v_1 \rightarrow \exists v_3 (E(v_3) \wedge v_3 \precsim v_2 \wedge \neg v_1 \circ v_3))$$

$$\text{(Fusion)} \qquad \forall X_1 (\exists v_1 X_1(v_1) \rightarrow \exists v_2 (FU(v_2, X_1))$$

The theory of Classical Mereology without Identity (CM^-) contains *MCM* and the following extra axiom:

$$\text{(NoZero)} \qquad \exists v_1 \exists v_2 (v_1 \not\precsim v_2) \rightarrow \neg \exists v_3 \neg (E(v_3))$$

The theory of Classical Mereology (*CM*) contains CM^- and the following extra axiom:

$$\text{(Anti-Symmetry)} \qquad \forall v_1 \forall v_2 (v_1 \precsim v_2 \wedge v_2 \precsim v_1 \rightarrow v_1 = v_2)$$

The minimal theory of Atomic Classical Mereology (*MACM*) / the theory of Atomic Classical Mereology without Identity (ACM^-) / the theory of Atomic Classical Mereology (*ACM*) contains *MCM*/CM^-/*CM* and the following extra axiom:

$$\text{(Atomicity)} \qquad \forall v_1 (E(v_1) \rightarrow \exists v_2 (At(v_2) \wedge v_2 \precsim v_1))$$

We have the minimal theory consisting of Transitivity, Supplementation and Fusion because these, as we will show in the next subsection, will be the core axioms that will be satisfied no matter whether we have vague existence, vague identity, or not, as we will show in the next subsection. The axioms NoZero and Anti-Symmetry are listed separately because these are the ones that do take a stand on whether there is vague existence/identity or not: the former disallows vague existence and the latter requires vague identity. An interesting observation is that the minimal theory *MCM* together with Anti-Symmetry forms a neutral system that is in between the classical theory of mereology[18] and the (second-order) theory of complete Boolean algebras, in the following sense:

Theorem 5.1. *CM* is equivalent to Tarski's system, which is the theory closed under the following two axioms:

$$\text{(Transitivity)} \qquad \forall v_1 \forall v_2 \forall v_3 (v_1 \precsim v_2 \wedge v_2 \precsim v_3 \rightarrow v_1 \precsim v_3)$$

$$\text{(UniqueFusionExistence)} \qquad \forall X_1 (\exists v_1 X_1(v_1) \rightarrow \exists ! v_2 (FU'(v_2, X_1))$$

where $FU'(v_2, X_1)$ is defined as: $FU'(v_2, X_1) = \forall v_3 (X_1(v_3) \rightarrow v_3 \precsim v_2) \wedge \forall v_4 (v_4 \precsim v_2 \rightarrow \exists v_5 (X_1(v_5) \wedge \exists v_6 (v_6 \precsim v_4 \wedge v_6 \precsim v_5)))$.

[18] By "the classical theory of mereology" I mean the theory that originates from Tarski's paper [25]. For a full development of Tarski's system, see [11].

Theorem 5.2. The (second-order) theory of complete Boolean algebra is equivalent to *MCM* plus Anti-symmetry plus the following axiom:

$$\text{(ZeroExistence)} \qquad \exists v_1 \neg E(v_1)$$

The proofs of these theorems are in the Appendix.

5.2 Atomic Boolean Models

We shall now define the atomic Boolean models. As we mentioned above, the domain of these models consists of functions from a pre-given set of mereological atoms S to a complete Boolean algebra B. But which of these functions shall we include in the domain exactly? For reasons I will explain in a moment there are at least two collections of functions from S to B that may reasonably form the domain of a model:

1. $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.
2. $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.

In the Appendix (Lemma D.2.1 and Lemma E.1.1) we will prove that in any atomic Boolean model, for any $f : S \to B$ in the domain, $\bigsqcup_{a \in S} f(a) = [\![E(f)]\!]$, the degree to which f exists. So the set M consists of functions that correspond to objects that exist to the degree 1. In our setting, to exist vaguely means to satisfy the existence predicate E to a degree that is in between 0 and 1. Therefore, atomic Boolean models with domain M have no room for vague existence at all. They will be used to show that under Boolean mereology, vague parthood does not entail vague existence, contrary to what many have argued, as there are Boolean models of vague parthood that are not models of vague existence. On the other hand, the set N consists of functions that correspond to objects that exist to any positive degree. Atomic Boolean models with domain N, therefore, have objects in their domains that exist vaguely. Under Boolean mereology, mereological vagueness can co-occur with existential vagueness, although not necessarily.

Definition 5.3. Let S be a set (of mereological atoms). Let B be a complete Boolean algebra. A B-valued *SE* ("Sharp-Existence") model on S, $\mathfrak{S}^B_S$, is a B-valued model for $\mathscr{L}_M$ with:

1. The domain $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.
2. For any $f_1, f_2 \in M$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_S} = \prod_{a \in S} (f_1(a) \Rightarrow f_2(a))$[19].

[19]For any p, q in a Boolean algebra B, $p \Rightarrow q = -p \sqcup q$.

A B-valued VE ("Vague-Existence") model on S, $\mathfrak{S}_V^B$, is a B-valued model for $\mathscr{L}_M$ with:

1. The domain $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.

2. For any $f_1, f_2 \in N$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}_V^B} = \prod_{a \in S} (f_1(a) \Rightarrow f_2(a))$.

In both kinds of models the values of parthood clauses are defined in the same way. Roughly, the degree to which an object is a part of another is defined as the degree of the sentence that every atom that is a part of the former is also a part of the latter.

Note that in defining these models we have omitted the definition of the values of identity clauses. This is because, depending on whether we want vague identity in our models or not, there are two different ways of defining identity in atomic Boolean models. The first way, which is given under the label "Vague-Identity", is to define identity in terms of the degree to which two objects share the same atoms. This is the way that is friendly to vague identity: it allows objects to be identical to each other to an intermediate degree. The second way, which is given under the label "Sharp-Identity", is to define identity "in the sharp way", that is, to define the degree to which two objects are identical as 1 when the corresponding functions are the same, and as 0 when the corresponding functions are different. This is the way, as you may expect, that is hostile to vague identity. Given two functions $f_1, f_2 : S \to B$:

(Vague-Identity) $\qquad [\![f_1 = f_2]\!] = \prod_{a \in S} (f_1(a) \Leftrightarrow f_2(a))$.

(Sharp-Identity) $\qquad$ If f_1 and f_2 are not the same, then $[\![f_1 = f_2]\!] = 0$.

We can freely combine Vague/Sharp-Identity with SE/VE models and get four different kinds of models, as listed in the following:

Definition 5.4. Let S be a set (of mereological atoms). Let B be a complete Boolean algebra. The B-valued $SEVI$ ("Sharp-Existence Vague-Identity") model on S, $\mathfrak{S}_{SV}^B$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.

2. For any $f_1, f_2 \in M$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}_{SV}^B} = \prod_{a \in S} (f_1(a) \Rightarrow f_2(a))$.

3. For any $f_1, f_2 \in M$, $[\![f_1 = f_2]\!]^{\mathfrak{S}_{SV}^B} = \prod_{a \in S} (f_1(a) \Leftrightarrow f_2(a))$.

The B-valued $SESI$ ("Sharp-Existence Sharp-Identity") model on S, $\mathfrak{S}^B_{SS}$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.
2. For any $f_1, f_2 \in M$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{SS}} = \bigsqcap_{a \in S}(f_1(a) \Rightarrow f_2(a))$.
3. For any $f_1, f_2 \in M$, if f_1 and f_2 are not the same, then $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{SS}} = 0$.

The B-valued $VEVI$ ("Vague-Existence Vague-Identity") model on S, $\mathfrak{S}^B_{VV}$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.
2. For any $f_1, f_2 \in N$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{VV}} = \bigsqcap_{a \in S}(f_1(a) \Rightarrow f_2(a))$.
3. For any $f_1, f_2 \in N$, $[\![f_1 = f_2]\!]^{\mathfrak{S}^B_{VV}} = \bigsqcap_{a \in S}(f_1(a) \Leftrightarrow f_2(a))$.

The B-valued $VESI$ ("Vague-Existence Sharp-Identity") model on S, $\mathfrak{S}^B_{VS}$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.
2. For any $f_1, f_2 \in N$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{VS}} = \bigsqcap_{a \in S}(f_1(a) \Rightarrow f_2(a))$.
3. For any $f_1, f_2 \in N$, if f_1 and f_2 are not the same, then $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{VS}} = 0$.

Assuming that B is larger than $\{0, 1\}$, all of the four different kinds of models are models of mereological vagueness, as it is easy to see that in all of the models there are objects that are part of one another to an intermediate degree. But they deliver different answers on whether there is vagueness in existence and/or on whether there is vagueness in identity. Just as in the case of existential vagueness, mereological vagueness can co-occur with vagueness in identity, but not necessarily.

In the rest of this section we will investigate which axioms of mereology hold in these four kinds of models. Most results will be simply stated here with the proofs in the Appendix.

As we have mentioned before, we formulate most axioms of mereology (all except Anti-Symmetry) without using the identity symbol. And hence whether these axioms hold or not in these models does not depend upon whether they are VI or SI. In fact,

Theorem 5.3. In any *SE* model, Transitivity, Supplementation, Fusion, Atomicity and NoZero all have value 1.

Theorem 5.4. In any VE model, Transitivity, Supplementation, Fusion and Atomicity all have value 1, but NoZero has value 0.

So the core theory of atomic classical mereology - and by that I mean the system *MACM* - is satisfied by all four kinds of models discussed here. Therefore, all four models can be legitimately considered models of atomic classical mereology. The difference between the VE and the *SE* models, of course, is that the axiom of NoZero does not hold in the VE models. This is, I believe, a somewhat unfortunate result for the supporters of vague existence. It means that if we allow objects that exist vaguely, then we will have to have the model believe that there is an empty object that is part of everything, even when there is more than one object. Under the standard conception of classical mereology, such an empty object is disallowed, because it is normally considered as philosophically unmotivated[20]. Nevertheless, it is not hard to see why there has to be tension between existential vagueness and the axiom of NoZero, in the current context. Assuming there is more than one object, then the axiom of NoZero has value 1 just in case every object f in the domain satisfies the existence predicate to the degree 1. So the axiom of NoZero literally leaves there to be no room for objects that exist to intermediate degrees. Proponents of existential vagueness has to sacrifice the axiom of NoZero.

Luckily, proponents of existential vagueness could argue that although the axiom of NoZero, in its current form, cannot be satisfied by models in which objects may exist vaguely, there is a satisfiable weaker meta-principle that is in the same spirit. The latter is the principle that there cannot be in the domain any object that is truly empty - that is, any object that satisfies the existence predicate to the degree 0. This has to be a principle in the meta-language because we simply do not have the expressive resources to state something of the form "x satisfies F to the degree p" in the object language. As it is easy to see, all VE models satisfy this meta-principle straightforwardly according to the definition of their domain N. Proponents of VE models could argue that although the VE models believe that there is an empty object, there isn't really an empty object in the domain of these models, and the latter is all we care about.[21]

[20]Although most people find the existence of an empty object philosophically unmotivated, there are some people who have provided ways to justify the existence of an empty object. Giraud [10] has construed it as a Meinongian object lacking all nuclear properties. Priest [21] has construed it as an Heideggerian nothing.

[21]This is an example of an intriguing and perhaps weird feature of Boolean-valued models. Some models could be such that an existential sentence is true in the model without there being a witness.

Moving on to the only axiom left - the axiom of Anti-Symmetry. As the readers might have expected, the holding or not of Anti-Symmetry in an atomic Boolean model is only associated with whether identity is defined in the vague way or in the sharp way in the model. Let us call a model a VI model if it is $SEVI$ or $VEVI$, and similarly a model a *SI* model if it is *SESI* or $VESI$. It can be shown that:

Theorem 5.5. In any VI model, Anti-Symmetry has value 1.

Theorem 5.6. In any *SI* model, Anti-Symmetry has value 0.

The opponents of vagueness in identity, therefore, have to sacrifice part of the standard package of classical mereology, just as the proponents of existential vagueness. In this case the sacrifice is the axiom of Anti-Symmetry. It is not hard to see why "Sharp-Identity" makes trouble for the holding of Anti-Symmetry: since there is mereological vagueness, there can be objects that are part of each other to an intermediate degree. Since their corresponding functions has to be different, "Sharp-Identity" insists that they are identical to the degree 0, and hence the degree to which they are part of each other is strictly greater than the degree to which they are identical, which causes the failure of Anti-Symmetry.

Just as the proponents of existential vagueness, there are, I believe, some ways for the opponents of vagueness in identity to argue back. They could say that, for example, in the context of mereology, there should really be two different notions of identity: one is the notion of mereological coincidence, and the other is the notion of strict/real identity. Two objects mereologically coincide - that is, are identical in the former sense - just in case they are indistinguishable in terms of mereological relations. On the other hand, two objects are strictly identical just in case they are indistinguishable in terms of *any* kind of properties or relations, mereological or not. And the key idea is that the equality symbol in the axiom of Anti-Symmetry should be interpreted as mereological coincidence instead of as strict identity: if two objects are part of one another, then they should be indistinguishable in terms of mereological relations, but saying that they should also be indistinguishable in terms of any relations seems like an overkill. In an atomic Boolean model, the degree to which two objects mereologically coincide should be defined according to "Vague-Identity", that is, as the degree to which two objects share the same atoms, and the degree to which two objects are strictly identical should be defined according to "Sharp-Identity", such that it can only be an extreme value. Since the relation that plays a role in Anti-Symmetry is mereological coincidence, we will have Anti-Symmetry hold in the models, and since strict identity is still defined traditionally, we also avoid the controversies surrounding vagueness in identity[22].

[22] A standard argument against vagueness in identity is Evans' Argument. See [8].

Below is a chart summarizing which axioms hold in each of the four kinds of atomic Boolean models:

	MACM	*MACM*+NoZero	*MACM*+Anti-Symmetry	*ACM*
SEVI	✓	✓	✓	✓
SESI	✓	✓	✗	✗
VEVI	✓	✗	✓	✗
VESI	✓	✗	✗	✗

Here's a summary of what we have achieved in this section. First, we have introduced a special kind of Boolean-valued models for mereology - the atomic Boolean models, and argued that they are intuitively motivated, given that the world is atomic. Second, we have used the *SEVI* models to exemplify our previous point that with Boolean degrees, we can have a degree-theoretic semantics that is compatible with the whole package of atomic classical mereology. Finally, we have used the atomic Boolean models to investigate the connection between mereological vagueness on the one hand and vagueness in existence and identity on the other hand. We have shown that contrary to many have argued, mereological vagueness entails neither existential vagueness nor vagueness in identity. With the four different kinds of atomic Boolean models, proponents of mereological vagueness can freely choose between having and not having vagueness in existence or identity: *SEVI* models for sharp existence plus vague identity, *SESI* models for sharp existence plus sharp identity, *VEVI* models for vague existence plus vague identity, and *VESI* models for vague existence plus sharp identity. There are, nevertheless, prices to be paid. Although all four models are models for the core theory of atomic classical mereology, the axiom of NoZero does not hold in the "Vague-Existence" models and the axiom of Anti-Symmetry does not hold in the "Sharp-Identity" models.

6 The Nature of Mereological Vagueness

Our investigation of Boolean mereology so far has been fruitful, but not all important questions on the topic of mereological vagueness have been properly addressed. For example, one essential question is: given that there is mereological vagueness, what is the source, or the nature of it? Is mereological vagueness a pure linguistic phenomenon, or is the world itself vague? Does the picture of Boolean mereology entail that mereological vagueness is semantic or ontological? In this section I intend to discuss these questions.

There are, I believe, two most commonly held answers to the question "What is the nature of mereological vagueness?". One option, which I will call "the semantic thesis" in the following, is to say that mereological vagueness has a semantic nature. The phenomenon exists because our linguistic practices are indeterminate, in the sense that they do not pin down the exact meanings of certain terms, including, perhaps, singluar names like "Tibbles". The linguistic rules that we have governing the name "Tibbles" does not pick out a unique referent for it. The world in itself, on the other hand, is perfectly precise, mereologically speaking: there is no indeterminacy in the mereological organization of the underlying reality. Mereological indeterminacy happens when we try to represent what the world is like using natural languages: if there were no language, or if natural languages were perfectly precise, there would be no indeterminacy in the parthood relation.

The other option, which I will call "the ontic thesis", is to say that mereological vagueness has an ontic, or worldly, nature. There is indeed indeterminacy in the mereological organization of the underlying reality. Regardless of the terms we use to represent them, ordinary objects in the world, like for example Tibbles the cat, are themselves vague, in the sense that their mereological constitution is indeterminate. Mereological vagueness is a feature of the world itself, not a feature of our languages.

Which one of the two theses do we have to adopt, as a Boolean mereologist? I believe that Boolean mereology, as the simple thesis that the relation of parthood should be modeled by Boolean degrees, is compatible with either thesis. Boolean mereology only says that sentences like "W is part of Tibbles" are true to an intermediate Boolean degree; it does not specify *why* these sentences are true to an intermediate Boolean degree. I will show below that the model-theoretic framework of Boolean-valued semantics can be applied to both theses and give rise to two distinctive views that have their unique advantages and disadvantages. I will call the view we get by combining the semantic thesis and Boolean-valued semantics "semantic Boolean mereology" and the view we get by combining the ontic thesis and Boolean-valued semantics "ontic Boolean mereology", and discuss them in turn in the following two subsections.

6.1 Semantic Boolean Mereology

The semantic thesis explains mereological indeterminacy in terms of linguistic indeterminacy and denies worldly indeterminacy. The most standard and commonly-held version of the view locates the indeterminacy at singular names like "Tibbles" or "Kilimanjaro". On this view, all there is in the world are objects with precise mereological boundaries. Names like "Tibbles" do not pick out a unique referent among the precise objects. Rather, there are multiple precise objects, located where Tibbles is, that are equally qualified candidates of being the referent of "Tibbles".

How does Boolean-valued semantics accommodate this view? To simplify our discussion, let us assume that the world is atomic and all that exists is (sharp) fusions of atoms. Let S be the collection of all atoms. Since all that exists is (sharp) fusions of atoms, the domain of our Boolean-valued model has to be the collection M' of all functions from S to $\{0,1\}$ except the one that takes all atoms to 0, where each function represents a fusion of atoms by being its characteristic function. As there are only precise objects in the domain, the identity symbol in the model can simply be interpreted as the sharp identity function on these objects. Now, since we want "Tibbles" to have no unique referent, "Tibbles" cannot be treated as an ordinary constant in the model. Rather, we need it to be the case that "Tibbles" indeterminately refers to multiple objects in the domain. In the context of Boolean-valued semantics, indeterminacy means having an intermediate truth value. So we want "Tibbles" to be interpreted in the model as a function from M' to B, which maps each object in the domain to the degree to which the name "Tibbles" refers to it. In other words, we will treat "Tibbles" semantically as if it were a unary predicate. Of course, "Tibbles" cannot be treated as if it were an arbitrary unary predicate: there are further constraints that the interpretation of "Tibbles" has to satisfy. In particular, the interpretation of "Tibbles" has to be such that the sentence $\exists! v(Tibbles(v))$ - there is exactly one Tibbles - has value 1. As a result, the values attributed to the objects by (the interpretation of) "Tibbles" has to form a maximal antichain in the Boolean algebra.

Let me spell out the above picture in more details, by constructing a concrete B-valued[23] model for the language consisting of "Tibbles", "W", and "is part of", $\mathfrak{M}'$, tailored to the needs of the standard semantic approach. Again, we assume that the world is atomic and all that exists is (sharp) fusions of atoms. Also, we assume, just for simplicity, that the name "W" picks out, instead of a whisker, an atom in the whisker that is about to fall off from the cat. The domain of the model, M', consists of all functions from S to $\{0,1\}$, except the one that takes all $a \in S$ to 0. That is, $M' = \{g : S \to B \mid \text{for any } a \in S, g(a) = 0 \text{ or } 1, \text{ and for some } b \in S, g(b) \neq 0\}$, which is equivalent to $\mathscr{P}(S)$ (the powerset of S) minus the empty set. Let the language $\mathscr{L}'$ be $\{t, w, \precsim\}$, where $\precsim$ is the binary relation of parthood, w is a constant playing the role of "W", and t is a unary predicate playing the role of "Tibbles". Since w is supposed to name an atom, the interpretation of w in $\mathfrak{M}'$ will be the characteristic function of a singleton subset $\{a\}$ of S. In other worlds, $[\![w]\!]^{\mathfrak{M}'} = g^a : S \to B$, where $a \in S$ and g^a takes a to 1 and every $b \neq a \in S$ to 0. The interpretation of $\precsim$ in $\mathfrak{M}'$ will be the function from $M' \times M' \to 2$ that corresponds to the subset relationship on $\mathscr{P}(S) \backslash \varnothing$. The interpretation of $=$ in $\mathfrak{M}'$ will be the "real" identity relation on M': for any $g, g' \in M$, $[\![g = g']\!] = 1$ if g and g' are the same and $[\![g = g']\!] = 0$ if g and

[23]Here we assume B is an arbitrary complete Boolean algebra.

g' are not the same. Finally, the interpretation of t in $\mathfrak{M}'$, $[\![t]\!]^{\mathfrak{M}'}$, will be a function from M' to B that satisfies the following conditions:

1. For any $g \neq g' \in M'$, $[\![t(g)]\!]^{\mathfrak{M}'} \sqcap [\![t(g')]\!]^{\mathfrak{M}'} = 0$.
2. $\bigsqcup_{g \in M'} [\![t(g)]\!]^{\mathfrak{M}'} = 1$.
3. For some $g \in M$ such that $[\![t(g)]\!]^{\mathfrak{M}'} \neq 0$, $g(a) = 1$, and for some $g' \in M$ such that $[\![t(g')]\!]^{\mathfrak{M}'} \neq 0$, $g(a) = 0$.

For every $g \in M'$, $[\![t(g)]\!]^{\mathfrak{M}'}$ is the degree to which t "refers to" g. $[\![t(g)]\!]^{\mathfrak{M}'} \neq 0$ means that g is a *possible*, or *permissible* referent of t. The third condition serves many purposes: first, it guarantees that there is more than one permissible referent of t; second, it means that w is part of some permissible referent of t yet is not part of some other permissible referent of t; and third, together with the first two conditions, it ensures that no g is the *determinate* referent of t, in the sense that $[\![t(g)]\!]^{\mathfrak{M}'} = 1$. The first two conditions also guarantee that $[\![\exists! v(t(v))]\!]^{\mathfrak{M}'} = 1$: it is true in $\mathfrak{M}'$ that there is exactly one t.

What is the degree to which w is part of t in $\mathfrak{M}'$? We want it to be an intermediate degree between 0 and 1, capturing the fact that it is indeterminate whether w is part of t. And the conditions we impose on the interpretation of t in $\mathfrak{M}'$ can indeed guarantee that. But there is a small complication. The sentence that $w \precsim t$ contains t syntactically as a constant, yet our model $\mathfrak{M}'$ treats t as a unary predicate. So we need to find some way to translate this sentence, or any sentence that contains t syntactically as a constant, to a sentence that contains t syntactically as a unary predicate. The trick we will use here is to translate any sentence of the form $\phi(t)$, which has t as a constant, to the sentence $\exists! v_i(t(v_i)) \wedge \forall v_j(t(v_j) \rightarrow \phi(v_j))$. It is easy to check that this translation recipe always preserves truth values for sentences involving constants. Moving on to the sentence under concern: (let $T' = \{g \in M' \mid [\![t(g)]\!]^{\mathfrak{M}'} \neq 0 \text{ and } g(a) = 0\}$)

$$\begin{aligned}
[\![w \precsim t]\!]^{\mathfrak{M}'} &= [\![\exists! v_i(t(v_i)) \wedge \forall v_j(t(v_j) \rightarrow (w \precsim v_j))]\!]^{\mathfrak{M}'} \\
&= [\![\forall v_j(t(v_j) \rightarrow (w \precsim v_j))]\!]^{\mathfrak{M}'} \\
&= \bigsqcap_{g \in M'} [\![t(g)]\!]^{\mathfrak{M}'} \Rightarrow g(a) \\
&= \bigsqcap_{g \in T'} -[\![t(g)]\!]^{\mathfrak{M}'} = -\bigsqcup_{g \in T'} [\![t(g)]\!]^{\mathfrak{M}'}
\end{aligned}$$

The three conditions we impose on the interpretation of t guarantees that $0 < \bigsqcup_{g \in T'} [\![t(g)]\!]^{\mathfrak{M}'} < 1$. Therefore, $0 < [\![w \precsim t]\!]^{\mathfrak{M}'} < 1$, which is exactly what we want.

Since $\mathfrak{M}'$ is a Boolean-valued model, all principles of classical logic will hold in it. Also, as it is easy to see that $\mathfrak{M}'$ restricted to the language of mereology $\mathscr{L}_M$ is isomorphic to the powerset model on S, the whole package of atomic classical mereology, by which I mean the system ACM, will hold in $\mathfrak{M}'$. One feature $\mathfrak{M}'$ worth mentioning is that $\mathfrak{M}$ is not a "witnessing" model, in the sense there are existential sentences whose truth value is strictly greater than that of any of its instances. For example, the sentence "something is Tibbles" will have value 1 in the model without any of its instances having value 1. But this is exactly what supporters of the semantic thesis would want: although they would agree that "Tibbles exists" is true, they would not identity any (sharp) object in the domain as uniquely identical to Tibbles.

Therefore, Boolean-valued semantics, as shown above, provides an elegant model theory for the semantic thesis. Under semantic Boolean mereology, the actual world that we live in is just like the model $\mathfrak{M}$ we constructed above. All there is are sharp objects, and the parthood relation that holds between them is also sharp. Mereological indeterminacy is grounded in the linguistic indeterminacy of terms like "Tibbles", which is further explained in terms of there being multiple objects in the domain to which the term applies to a degree larger than 0.

The standard model-theoretic framework that accompanies the semantic thesis is supervaluation semantics[24]. A supervaluation model consists of a fixed domain of objects[25] and multiple permissible precisifications. Each precisification can be understood as a two-valued model with the given domain. A sentence is (super)true if it is true in all precisifications, (super)false if false in all precisifications, and neither (super)true nor (super)false if otherwise. On cases like Tibbles, each permissible precisification assigns to "Tibbles" a different object in the domain as its referent. "W is part of Tibbles", in the intended model, will be a sentence that is neither (super)true nor (super)false. A supervaluation model is actually a special case of Boolean-valued models like $\mathfrak{M}'$. Let $\mathfrak{S}$ be a supervaluation model for $\mathscr{L}'$ with domain D and precisifications $\{\mathfrak{A}_i \mid i \in I\}$, where in each $\mathfrak{A}_i$, $[\![w]\!]^{\mathfrak{A}_i} = a \in D$ and $[\![t]\!]^{\mathfrak{A}_i} = a_i \in D$. We can transform $\mathfrak{S}$ to a $\mathscr{P}(I)$-valued Boolean model $\mathfrak{M}^{\mathfrak{S}}$ with domain D as follows:

1. $[\![w]\!]^{\mathfrak{M}^{\mathfrak{S}}} = a$.

2. For any $b \in D$, $[\![t(b)]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{M}_i \models t = b\}$.

[24] As in, for example, [9].

[25] Sometimes supervaluationism is used on cases where it is indeterminate what the domain of quantification is. One example are cases of quantum indeterminacy (see [4] or [15]). On cases of mereological indeterminacy, nevertheless, it is usually safe to assume that the domain of quantification is determinate.

3. For any $b, c \in D$, $[\![b \precsim c]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{M}_{\mathrm{i}} \models b \precsim c]\!]$.

Using the translation recipe we introduced above, $[\![w \precsim t]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{A}_{\mathrm{i}} \models w \precsim t\}$ will be a proper non-empty subset of I, as there are precisifications in which w is part of t and ones in which w is not part of t. That $w \precsim t$, therefore has an intermediate truth value in $\mathfrak{M}^{\mathfrak{S}}$, which corresponds to that it is neither (super)true nor (super)false in $\mathfrak{S}$. Although mathematically speaking, transforming a supervaluation model into a Boolean-valued model makes no significant difference, from a philosophical perspective such a transformation brings upon a number of benefits. Since supervaluation models now become Boolean-degree-theoretic, they can enjoy all the advantages that the Boolean semantics has, as discussed in the previous sections: being truth-functional, having distinct comparable borderline statuses, having incomparable borderline statuses etc.

Semantic Boolean mereology is an attractive story and enjoys many theoretic advantages. For example, some people ([7]) have argued that vagueness should be a uniform phenomenon, in the sense that different types of vagueness should have the same nature: they are either all semantic or all ontic. Since there are strong arguments for vagueness in properties (like the property of being bald) being a semantic phenomenon, mereological vagueness should be theorized as a semantic phenomenon as well. To me, the biggest advantage held by semantic Boolean mereology is that it naturally comes with a solution to the notorious problem of the many (see, for example, [17]). As long as we accept classical mereology, the principle of fusion existence will generate a great number of distinct objects that heavily overlap with each other, all located where Tibbles is. There are, then, two seemingly contradictory intuitions. The first intuition is that there should be only one referent of "Tibbles", instead of many. The second intuition is that since these objects only have minute differences - say, only in whether it has an atom on the periphery of Tibbles like w, no one among them seems to have a better claim to be the referent of "Tibbles" than others. It is not hard to see how semantic Boolean mereology resolves this apparent contradiction. Under semantic Boolean mereology, the candidate referents are all such that it is indeterminate whether they are the referent of "Tibbles", in the sense that "Tibbles" refer to them to an intermediate degree, and none of these degrees are strictly higher or lower than any one of the others. This captures the second intuition. Meanwhile, "there is only one Tibbles" always has value 1 in the intended models, which corresponds to the first intuition.

Despite its advantages, semantic Boolean mereology also has some problems. An immediate consequence of semantic Boolean mereology is that the majority of names of ordinary objects - "Tibbles", "Kilimanjaro", "Marie Curie", "Earth", "Eiffel Tower", etc. - do not refer successfully, in the sense that they do not fix a unique referent. This is a bizarre consequence. It means that our ordinary methods of identifying and naming objects almost always fail, even under the best possible circumstances. The level of referential ambiguity displayed in the scenario in which I point to the only furry creature in the room and say "this is Tibbles" is the same as that displayed in the scenario in which I point to a corner where there are three men and say "this is John". If the foundation of our theory of meaning, as many have proposed, is that names designate objects, then that foundation is based on an impossible idealization.

Also, although semantic Boolean mereology is not completely incompatible with the existence of ordinary objects, ordinary objects under semantic Boolean mereology, in some sense, are ontologically shallow. Let us consider Tibbles the cat. Under this theory, the sentence that "Tibbles exists" is true to the degree 1, and in this sense ordinary objects like Tibbles do exist. But since all there is in the domain of the intended models are objects with precise mereological boundaries, there is no existing object that is really, or determinately, identical to Tibbles. In other words, there is no object x such that "Determinately x is Tibbles" is true to the degree 1. So Tibbles, in a certain sense, does not really exist. This is, I believe, not quite in line with our common-sense conception of Tibbles' existence: normally we would think that there exists a cat in the world that truly is Tibbles.

6.2 Ontic Boolean Mereology

Unlike the semantic thesis, the ontic thesis holds that there are indeed objects in the world that are vague in their mereological organization, and names of these objects refer to them in the standard, determinate way. Under the context of Boolean semantics, this is to say that there are objects in the domain such that they stand in the parthood relation with some other objects to intermediate Boolean degrees, which are (unique) referents of some constants. The intended models for ontic Boolean mereology, then, are models like the atomic Boolean models. Take, for example, the $SEVI$ model $\mathfrak{S}^B_{SV}$ for $\mathscr{L}_M$. We may extend $\mathfrak{S}^B_{SV}$ to a model for $\mathscr{L}'$ by letting w denote some $g^a \in M$ such that $a \in S$ and g^a takes a to 1 and every $b \neq a \in S$ to 0, and t denote some $f \in M \backslash M'$ such that $f(a)$ is some intermediate value between 0 and 1. In other words, w (determinately) denotes (the characteristic function of) some atom and t (determinately) denotes (the characteristic function of) a vague object whose value distribution on atoms, especially on w, involves intermediate values.

There are, then, two core differences between semantic Boolean mereology and ontic Boolean mereology. The first difference is that the domain of an intended model for semantic Boolean mereology contains only sharp objects, whereas the domain of an intended model for ontic Boolean mereology contains both sharp objects and vague objects. The second difference is that simple names like "Tibbles" under ontic Boolean mereology are interpreted normally as constants and have determinate referents, whereas under semantic Boolean mereology they are interpreted syntactically as unary predicates and have multiple indeterminate referents.

In my opinion, Boolean-valued semantics provides the best model-theoretic framework for proponents of the ontic thesis. The two alternative semantics framework, in comparison to Boolean-valued semantics, both have serious problems. The first alternative is the fuzzy-valued model theory, and in section Four I have already argued in great length why it is less favorable than Boolean model theory, when applying to mereological indeterminacy. The second alternative is supervaluation model theory. But unlike in the semantic case, the combination of supervaluation semantics and the ontic thesis (see [1]), in my opinion, yields an awkward theory, which I will call "ontic supervaluationism" in the following. Under ontic supervaluationism, there are multiple distinct "precisificatons" *of the underlying reality* that are used to explain mereological vagueness. Although the model-theoretic techniques employed in this view is basically identical to that in the semantic case, from the philosophical perspective ontic supervaluationism feels much more unnatural and faces more difficult questions, compared to its semantic counterpart. For example, in the case of semantic supervaluationism, we have a fairly good understanding of what a "precisification" is: it is a total interpretation function that is consistent with how we use terms like "Tibbles" in languages. But what is, or can be, a "precisification" *of the reality*, in the case of ontic supervaluationism? It cannot be language or mind dependent, as it is supposed to capture a feature of the world, so is it something that exists out there? What is its ontological status? If it is like a possible world that exists along side our world, why is the vagueness of the objects in our world grounded in these things? Also, following the ontic thesis, the referent of the name "Tibbles" needs to be an object that exists in the actual world, but somehow it also has to be a different object in each of these precisifications - how exactly can we reconcile these claims? I do not see an easy answer for any of these questions, and therefore I think that supervaluation semantics is not really a viable option for supporters of the ontic thesis.

Just like semantic Boolean mereology, or perhaps any philosophical theory, ontic Boolean mereology has its advantages and disadvantages. Its biggest advantage is that it overcomes the two difficulties held by semantic Boolean mereology, as presented in the previous subsection. Under ontic Boolean mereology, we are not stuck with a vast scale of referential failure. Also for ordinary objects like Tibbles, we will have something existing in the domain that is determinately Tibbles, so the existence of Tibbles is not ontologically shallow. The biggest problem plaguing ontic Boolean mereology, on the other hand, is the problem of many. Again, if we accept the principle of fusion existence, there will be a number of distinct vague objects with minute differences, all located at where Tibbles is. Now, ontic semantic mereology claims that there is among them a unique referent of “Tibbles”, but which one of these objects should be the unique referent? Consider, for example, the model $\mathfrak{S}_{SV}^{B}$. Every function in M corresponds to an object in the world, and as long as B is large enough, there can be many functions f in M that (1) has the same value on every other atom except $a \in S$, and (2) has an intermediate value on a (let a be the referent of w in $\mathfrak{S}_{SV}^{B}$). The difficult question seems to be: what makes one of them a better candidate for being the referent of t than others?

Note that this is a problem that troubles all supporters of the ontic thesis, not just supporters of ontic Boolean mereology. Ontic fuzzy mereology and ontic supervaluationism faces this problem in very similar manners. In my opinion, the simplest, and the best way for the ontic Boolean mereologists to respond to the problem of many is to reject the principle of fusion existence and embrace an ontology that is less well-populated. In a model like $\mathfrak{S}_{SV}^{B}$, for example, they could say that not all functions in M corresponds to an object existing in the world. Rather, only one of the many possible profiles of value distribution on the atoms relevant to Tibbles actually corresponds to an existing (ordinary) object - a cat, in particular, and that is the unique referent of “Tibbles”. The difficult question they would face then, which I will call the “special condition question”, is “What’s special about this particular value profile, compared to the others, that makes it a profile of an object?”. At this point, there are two kinds of responses on the table. The first response is to suggest that there is some kind of naturalness condition satisfied by this value profile, perhaps in terms of contact and adhesion, that is responsible for its “objecthood” . The second response is to claim that it is simply a piece of brute fact that this particular value profile corresponds to an object. And in general, there are just brute facts true of the world we live in that some Boolean value profiles correspond to actually existing (ordinary) objects whereas others do not[26].

Does this mean that ontic Boolean mereologists have to completely forsake classical mereology? Not necessarily. What they have to deny is that classical mereology - the principle of fusion existence, in particular, holds on *ordinary objects* like cats, but they could still say that it holds on more fundamental and abstract entities like spacial-temporal regions. They could hold that, for example, any Boolean profile on spacial-temporal points(atoms) corresponds to a spacial-temporal region that is part of the ontology, but only one of the (relevant) special-temporal regions is occupied by a cat-like entity, which is Tibbles the cat. Of

[26]Note that when facing a similar many-valued version of the problem of many, the ontic fuzzy mereologists also typically tend to choose one of the two possible responses discussed here to the special condition question. Nicholas Smith, for example, uses the first kind of response in [24]. Peter van Inwagen uses the second kind of response in [26].

course, what they would have to answer, then, is a slightly different version of the special condition question, perhaps along the lines of "What's special about this particular value profile, compared to the others, that makes it a profile of an ordinary, cat-like object?", and they could again adopt one of the two potential responses. The point here is just that ontic Boolean mereologists do have the freedom to choose between a sparse ontology and a sparser ontology, and between completely and partially denying the principle of fusion existence.

Acknowledgments

For their helpful feedback on earlier drafts of this paper, I would like to thank Vann McGee, Agustín Rayo and Stephen Yablo. I would also like to thank the editor and two anonymous referees of this journal. Additionally, I am grateful to the participants of the WIP seminar at MIT, especially to Jack Spencer.

References

[1] Akiba, K. (2004). Vagueness in the world. *Noûs*, 38(3):407–429.

[2] Barnes, E. and Williams, J. R. G. (2009). Vague parts and vague identity. *Pacific Philosophical Quarterly*, 90(2):176–187.

[3] Bell, J. L. (2011). *Set theory: Boolean-valued models and independence proofs*. Oxford University Press.

[4] Calosi, C. and Wilson, J. (2019). Quantum metaphysical indeterminacy. *Philosophical Studies*, 176(10):2599–2627.

[5] Cook, M. (1986). Indeterminacy of identity. *Analysis*, 46(4):179–186.

[6] Donnelly, M. (2009). Mereological vagueness and existential vagueness. *Synthese*, 168(1):53–79.

[7] Eklund, M. (2008). Deconstructing ontological vagueness. *Canadian Journal of Philosophy*, 38(1):117–140.

[8] Evans, G. (1978). Can there be vague objects. *Analysis*, 38(4):208.

[9] Fine, K. (1975). Vagueness, truth and logic. *Synthese*, pages 265–300.

[10] Giraud, T. (2013). An abstract mereology for meinongian objects. *Journal of Philosophical Studies*, 6(25):177–210.

[11] Gruszczyński, R. and Pietruszczak, A. (2008). Full development of Tarski's geometry of solids. *Bulletin of Symbolic Logic*, 14(4):481–540.

[12] Hovda, P. (2009). What is classical mereology? *Journal of Philosophical Logic*, 38(1):55–82.

[13] Keefe, R. (2000). *Theories of vagueness*. Cambridge University Press.

[14] Lewis, D. (1986). *On the plurality of worlds*. Blackwell Oxford.

[15] Mariani, C., Michels, R., and Torrengo, G. (2021). Plural metaphysical supervaluationism. *Inquiry*, pages 1–38.

[16] McGee, V. and McLaughlin, B. (1995). Distinctions without a difference. *The Southern Journal of Philosophy*, 33(Supplement):203–251.

[17] McGee, V. and McLaughlin, B. (2000). The lessons of the many. *Philosophical Topics*, 28(1):129–151.

[18] McGee, V. and McLaughlin, B. (Forthcoming). *Terrestrial logic*. Oxford University Press.

[19] Merricks, T. (2005). Composition and vagueness. *Mind*, 114(455):615–637.

[20] Morreau, M. (2002). What vague objects are like. *The Journal of Philosophy*, 99(7):333–361.

[21] Priest, G. (2014). Much ado about nothing. *The Australasian Journal of Logic*, 11(2).

[22] Rasiowa, H. (1963). *The mathematics of metamathematics*.

[23] Sainsbury, R. (1989). What is a vague object? *Analysis*, 49(3):99–103.

[24] Smith, N. J. (2005). A plea for things that are not quite all there: or, is there a problem about vague composition and vague existence? *The Journal of philosophy*, 102(8):381–421.

[25] Tarski, A. (1956). Foundations of the geometry of solids. *Logic, semantics, metamathematics*, 52:24–29.

[26] Van Inwagen, P. (1990). *Material beings*. Cornell University Press.

[27] Weatherson, B. (2003). Many many problems. *The Philosophical Quarterly*, 53(213):481–501.

[28] Williamson, T. (2002). *Vagueness*. Routledge.

[29] Wu, X. (2022). *Boolean-valued models and their applications*. PhD thesis, Massachusetts Institute of Technology.

Appendices

A Preliminaries on Boolean Model Theory

Definition A.1. Let $\mathscr{L}$ be an arbitrary first-order/second-order language. For simplicity, we assume that $\mathscr{L}$ has no function symbols/variables, but only relation symbols/variables, individual constants/variables.[27] Let B be a complete Boolean algebra. A B-valued model[28] $\mathfrak{A}$ for the language $\mathscr{L}$ consists of:

1. A universe A of elements;
2. The B-value of the identity symbol: a function $[\![=]\!]^{\mathfrak{A}} : A^2 \to B$;
3. The B-values of the relation symbols: (let P be a n-ary relation) $[\![P]\!]^{\mathfrak{A}} : A^n \to B$;
4. The B-values of the constant symbols: (let c be a constant) $[\![c]\!]^{\mathfrak{A}} \in A$.

And it needs to satisfy:

1. For the B-value of the identity symbol[29]: for any $a_1, a_2, a_3 \in A$

$$[\![a_1 = a_1]\!]^{\mathfrak{A}} = 1_B \tag{1}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} = [\![a_2 = a_1]\!]^{\mathfrak{A}} \tag{2}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} \sqcap [\![a_2 = a_3]\!]^{\mathfrak{A}} \leqslant [\![a_1 = a_3]\!]^{\mathfrak{A}} \tag{3}$$

2. For the B-value of relation symbols: let P be an n-ary relation; for any $\langle a_1, ..., a_n\rangle, \langle b_1, ..., b_n\rangle \in A^n$,

$$[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}} \sqcap \Big(\prod_{1 \leqslant i \leqslant n} [\![a_i = b_i]\!]^{\mathfrak{A}} \Big) \leqslant [\![P(b_1, ..., b_n)]\!]^{\mathfrak{A}} \tag{4}$$

[27] Our theory can be easily generalized to first order languages with function symbols, as functions can always be treated as relations that satisfy special conditions.

[28] See [29] for a detailed model theory on Boolean-valued models.

[29] Here and in the following, when the context is clear, we use $[\![a_i = a_j]\!]^{\mathfrak{A}}$ to abbreviate $[\![=]\!]^{\mathfrak{A}}(a_i, a_j)$, and similarly for cases of the relation symbols.

Definition A.2. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. For any $n \in \omega$, we define D^n_A as the following set: $D^n_A = \{R : A^n \to B \mid \text{for any } \langle a_1,...,a_n\rangle, \langle b_1,...,b_n\rangle \in A^n, R(a_1,...,a_n) \sqcap (\prod_{1 \leqslant i \leqslant n} [\![a_i = b_i]\!]^{\mathfrak{A}}) \leqslant R(b_1,...,b_n)\}$. We call the D^n_A's the *second-order domains* of $\mathfrak{A}$. For each $n \in \omega$, we call D^n_A the n-ary second-order domain of $\mathfrak{A}$.

Given a B-valued model $\mathfrak{A}$ for $\mathscr{L}$, we define satisfaction in $\mathfrak{A}$ as follows. Let *Var* be the set of all variables. An assignment s on $\mathfrak{A}$ is a function with domain *Var* such that (1) for any individual variable v_i, $s(v_i) \in A$, and (2) for any relation variable X_i of arity n, $s(X_i) \in D^n_A$.[30] We define the value of a term/an atomic open formula in $\mathfrak{A}$ under assignment s in the standard way. For complex formulas,

$$[\![\neg\phi]\!]^{\mathfrak{A}}[s] = -[\![\phi]\!]^{\mathfrak{A}}[s].$$

$$[\![\phi \wedge \psi]\!]^{\mathfrak{A}}[s] = [\![\phi]\!]^{\mathfrak{A}}[s] \sqcap [\![\psi]\!]^{\mathfrak{A}}[s].$$

$$[\![\exists v_i \phi]\!]^{\mathfrak{A}}[s] = \bigsqcup_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[s(v_i/a)].$$

$$[\![\exists X_i \phi]\!]^{\mathfrak{A}}[s] = \bigsqcup_{R \in D^n_A} [\![\phi]\!]^{\mathfrak{A}}[s(X_i/R)].$$

The values of the quantified formulas are well-defined as B is complete. We say that ϕ is a first-order formula when ϕ has no second order variables.

Theorem A.1. Let $\mathfrak{A}$ be a B-valued model for $\mathscr{L}$. For any formula $\phi(v_1,...,v_n)$ in $\mathscr{L}$, any assignments s, s' on $\mathfrak{A}$,

$$[\![\phi(s(v_1),...,s(v_n))]\!]^{\mathfrak{A}} \sqcap (\prod_{1 \leqslant i \leqslant n} [\![s(v_i) = s'(v_i)]\!]^{\mathfrak{A}}) \leqslant [\![\phi(s'(v_1),...,s'(v_n))]\!]^{\mathfrak{A}}$$

Proof. By a straightforward induction on the complexity of $\phi(v_1,...,v_n)$. □

B Soundness and Completeness

Definition B.1. Let T be a theory in a language $\mathscr{L}$. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is a model of T just in case for any $\phi \in T$, $[\![\phi]\!]^{\mathfrak{A}} = 1_B$.

Definition B.2. Let T be a theory and ϕ be a sentence in a language $\mathscr{L}$. ϕ is a Boolean-consequence of T, in symbols, $T \models_B \phi$ just in case for any Boolean valued model $\mathfrak{A}$, if $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ is a model of ϕ.

In the rest of this section we assume that $\mathscr{L}$ is a first-order language.

[30] In the case when $\mathscr{L}$ is a first-order language, this line can simply be ignored, for obvious reasons. And similarly for 2(c), 3(f) and 3(g) below.

Theorem B.1. Let T be a theory and ϕ be a sentence in $\mathscr{L}$. If $T \vdash \phi$, then $T \models_B \phi$.

Proof. See [22] or [29] for a detailed proof. □

Corollary B.1.1. Let ϕ be a theorem of first order logic. Then in any Boolean valued model $\mathfrak{A}$, $[\![\phi]\!]^{\mathfrak{A}} = 1$.

Theorem B.2. Let T be a theory in $\mathscr{L}$. T is consistent if and only if for some complete Boolean Algebra B, T has a B-valued model $\mathfrak{A}$.

Proof. See [22] or [29] for a detailed proof. □

Corollary B.2.1. Let B be any complete Boolean algebra. A theory T has a B-valued model just in case every finite subset of T has a B-valued model.

Theorem B.3. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. If $T \models_B \phi$, then $T \vdash \phi$.

Corollary B.3.1. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. $T \models_B \phi$ if and only if $T \vdash \phi$.

C Equivalence Between Systems

In this section we prove the two promised theorems in Section Five (Theorem 5.1 and Theorem 5.2).

Theorem C.1. *CM* is equivalent to Tarski's system, which is the theory closed under the following two axioms:

(Transitivity)	$\forall v_1 \forall v_2 \forall v_3 (v_1 \precsim v_2 \wedge v_2 \precsim v_3 \rightarrow v_1 \precsim v_3)$
(UniqueFusionExistence)	$\forall X_1 (\exists v_1 X_1(v_1) \rightarrow \exists! v_2 (FU'(v_2, X_1))$

Proof. We first show that *CM* entails Tarski's system. (Transitivity) is already in *CM*. For (UniqueFusionExistence), let X_1 be such that $\exists v_1 X_1(v_1)$. By (Fusion), $\exists v_2 (FU(v_2, X_1))$. Let $v_3 \leqslant v_2$. If $E(v_3)$, then we are done. Suppose $\neg E(v_3)$. By (NoZero), $\forall v_4 \forall v_5 (v_4 \precsim v_5)$. Hence trivially $v_3 \precsim v_3$ and $v_3 \precsim v_1$.

For the other direction, we can just use the standard argument that these axioms are all theorems of Tarski's system. See, for example, [12].

□

Theorem C.2. The (second-order) theory of complete Boolean algebra (*CBA*) is equivalent to *MCM* plus Anti-symmetry plus the following axiom:

(ZeroExistence)	$\exists v_1 \neg E(v_1)$

Proof. For the direction that the latter system entails *CBA*, it suffices to show that (Reflexivity), (SupremumExistence), (Complementation) and (Distribution) are all theorems of the latter system. See [12] for proofs. For the other direction, the only axiom worth mentioning is (Fusion). We will show that if $Sup(v_1, X_1)$, then $FU(v_1, X_1)$. That v_1 is a upper bound of X_1 is obviously the case. We only need to show that $\forall v_2(v_2 \precsim v_1 \wedge E(v_2) \rightarrow \exists v_3(X_1(v_3) \wedge v_3 \circ v_2))$. Suppose the antecedent. Assume for reductio that $\forall v_4(X(v_4) \rightarrow v_2 \sqcap v_4 = 0)$. Then by infinite distribution, $v_2 \sqcap v_1 = 0$. Since $E(v_2)$, $v_2 \neq 0$. Hence $v_2 \not\precsim v_1$. Contradiction.

□

D The *SE* Models

In this section we prove the following result:

Theorem D.1. In any *SE* model, Transitivity, Supplementation, Fusion, Atomicity and NoZero all have value 1.

Theorem D.2 (Transitivity)**.** $\mathfrak{S}_S^B \models \forall v_1 \forall v_2 \forall v_3(v_1 \precsim v_2 \wedge v_2 \precsim v_3 \rightarrow v_1 \precsim v_3)$.

Lemma D.2.1. For any $f \in M$, $[\![E(f)]\!] = \bigsqcup_{a \in S} f(a) = 1$.[31]

Proof. $[\![E(f)]\!] = [\![\exists v_2(\neg f \precsim v_2)]\!] = \bigsqcup_{g \in M} \bigsqcup_{a \in S} f(a) \sqcap -g(a)$. We want to show that $\bigsqcup_{g \in M} \bigsqcup_{a \in S} f(a) \sqcap -g(a) = \bigsqcup_{a \in S} f(a)$. For any $a \in S$, let g^a be the function from S to B that takes a to 1 and every $b \neq a$ to 0. Obviously $g^a \in M$. Pick some $a \in S$, then it is easy to see for any $b \neq a \in S$, $f(a) \leqslant \bigsqcup_{c \in S} f(c) \sqcap -g^b(c)$. Hence $f(a) \leqslant [\![E(f)]\!]$. For the other direction, pick some $g \in M$. Obviously $\bigsqcup_{a \in S} f(a) \sqcap -g(a) \leqslant \bigsqcup_{a \in S} f(a)$.

□

Lemma D.2.2. For any $f_1, f_2 \in M$, $[\![f_1 \circ f_2]\!] = \bigsqcup_{a \in S} f_1(a) \sqcap f_2(a)$.

Proof. By definition, $[\![f_1 \circ f_2]\!] = [\![\exists v_3(E(v_3) \wedge v_3 \precsim f_1 \wedge v_3 \precsim f_2)]\!]$. Since every $g \in M$ is such that $[\![E(g)]\!] = 1$, $[\![f_1 \circ f_2]\!] = \bigsqcup_{g \in M} [\![g \precsim f_1]\!] \sqcap [\![g \precsim f_2]\!] = \bigsqcup_{g \in M} \bigsqcap_{a \in S} g(a) \Rightarrow (f_1(a) \sqcap f_2(a))$. We will show that this is equal to $\bigsqcup_{a \in S} f_1(a) \sqcap f_2(a) = p$.

For the $\leqslant$ direction: Fix $g \in M$. Since $\bigsqcup_{a \in S} g(a) = 1$, $\bigsqcap_{a \in S} g(a) \Rightarrow (f_1(a) \sqcap f_2(a)) = \bigsqcap_{a \in S} -g(a) \sqcup (f_1(a) \sqcap f_2(a)) \leqslant \bigsqcap_{a \in S} -g(a) \sqcup p = 0 \sqcup p = p$.

For the $\geqslant$ direction: Fix $a \in S$. Then it is easy to see that $f_1(a) = [\![g^a \precsim f_1]\!]$, and similarly $f_2(a) = [\![g^a \precsim f_2]\!]$. Hence $f_1(a) \sqcap f_1(a) \leqslant [\![f_1 \circ f_2]\!]$.

□

[31] We omit the superscripts when the context is clear.

Theorem D.3 (Supplementation)**.** $\mathfrak{S}_S^B \models \forall v_1 \forall v_2 (v_2 \not\precsim v_1 \to \exists v_3 (E(v_3) \wedge v_3 \precsim v_2 \wedge \neg v_1 \circ v_3))$.

Proof. Let $f_1, f_2 \in M$. Since every $g \in M$ is such that $[\![E(g)]\!] = 1$, we just need to show that $-[\![f_2 \precsim f_1]\!] \leqslant \bigsqcup_{g \in M} [\![g \precsim f_2]\!] \sqcap -[\![g \circ f_1]\!]$. $-[\![f_2 \precsim f_1]\!] = \bigsqcup_{a \in S} f_2(a) \sqcap -f_1(a)$. Fix some $a \in S$. $[\![g^a \precsim f_1]\!] = f_2(a)$. By the previous lemma, $-[\![g^a \circ f_1]\!] = -(\bigsqcup_{b \in S} g^a(b) \sqcap f_1(b)) = -f_1(b)$.

□

Theorem D.4 (Fusion)**.** $\mathfrak{S}_S^B \models \forall X_1 (\exists v_1 X_1(v_1) \to \exists v_2 (FU(v_2, X_1)))$.

Proof. We will show that for any $R \in D_M^1$, $[\![\exists v_1 R(v_1) \to \exists v_2 (FU(v_2, R))]\!] = 1$. That is, $q = \bigsqcup_{t \in M} R(t) \leqslant [\![\exists v_1 (\forall v_2 (R(v_2) \to v_2 \precsim v_1) \wedge \forall v_3 (v_3 \precsim v_1 \wedge E(v_3) \to \exists v_4 (R(v_4) \wedge v_3 \circ v_4)))]\!] = \bigsqcup_{f \in M} ((\bigsqcap_{g \in M} R(g) \Rightarrow [\![g \precsim f]\!]) \sqcap (\bigsqcap_{h \in M} ([\![h \precsim f]\!] \Rightarrow (\bigsqcup_{s \in M} R(s) \sqcap [\![h \circ s]\!]))))$.

We define $f^R \in M$ as follows: pick some particular $a \in S$, let $f^R(a) = (\bigsqcup_{g \in M} R(g) \sqcap g(a)) \sqcup -q$. For any $b \neq a \in S$, let $f^R(b) = \bigsqcup_{g \in M} R(g) \sqcap g(b)$.

We first show that f^R is indeed in M, i.e. $\bigsqcup_{c \in S} f^R(c) = 1$:

$$\begin{aligned}
\bigsqcup_{c \in S} f^R(c) &= (\bigsqcup_{b \neq a \in S} f^R(b)) \sqcup f^R(a) \\
&= (\bigsqcup_{b \neq a \in S} \bigsqcup_{g \in M} R(g) \sqcap g(b)) \sqcup ((\bigsqcup_{g \in M} R(g) \sqcap g(a)) \sqcup -q) \\
&= (\bigsqcup_{c \in S} \bigsqcup_{g \in M} R(g) \sqcap g(c)) \sqcup -q \\
&= (\bigsqcup_{g \in M} R(g) \sqcap \bigsqcup_{c \in S} g(c)) \sqcup -q = (q \sqcap 1) \sqcup -q = 1
\end{aligned}$$

Now we show that $\bigsqcap_{g \in M} R(g) \Rightarrow [\![g \precsim f^R]\!] = \bigsqcap_{g \in M} R(g) \Rightarrow (\bigsqcap_{c \in S} g(c) \Rightarrow f^R(c)) = 1$. Pick any $g \in M$. $R(g) \Rightarrow (\bigsqcap_{c \in S} g(c) \Rightarrow f^R(c)) = -R(g) \sqcup ((\bigsqcap_{c \neq a} -g(c) \sqcup f^R(c)) \sqcap (-g(a) \sqcup f^R(a))) = (\bigsqcap_{c \neq a} -R(g) \sqcup -g(c) \sqcup f^R(c)) \sqcap (-R(g) \sqcup -g(a) \sqcup f^R(a))$. $\bigsqcap_{c \neq a} -R(g) \sqcup -g(c) \sqcup f^R(c) = \bigsqcap_{c \neq a} -R(g) \sqcup -g(c) \sqcup (\bigsqcup_{h \in M} R(h) \sqcap h(c)) \geqslant \bigsqcap_{c \neq a} -R(g) \sqcup -g(c) \sqcup (-R(g) \sqcap g(c)) = 1$. $-R(g) \sqcup -g(a) \sqcup f^R(a) = -R(g) \sqcup -g(a) \sqcup (\bigsqcup_{h \in M} R(h) \sqcap h(c)) \sqcup -q = 1$.

We next show that $q \leqslant \bigsqcap_{h \in M} ([\![h \precsim f^R]\!] \Rightarrow (\bigsqcup_{s \in M} R(s) \sqcap [\![h \circ s]\!]))$. Fix any $h \in M$. We want to show that $q \leqslant (\bigsqcup_{c \in S} h(c) \sqcap -f^R(c)) \sqcup (\bigsqcup_{d \in S} \bigsqcup_{s \in M} R(s) \sqcap s(d) \sqcap h(d)) = p$. Now it is easy to see that $p = p_1 \sqcup p_2$, where $p_1 = (\bigsqcup_{c \neq a} h(c) \sqcap -f^R(c)) \sqcup (\bigsqcup_{d \neq a} \bigsqcup_{s \in M} R(s) \sqcap s(d) \sqcap h(d))$ and $p_2 = (h(a) \sqcap -f^R(a)) \sqcup (\bigsqcup_{s \in M} R(s) \sqcap s(a) \sqcap h(a))$. But $p_1 = (\bigsqcup_{c \neq a} h(c) \sqcap -f^R(c)) \sqcup (\bigsqcup_{d \neq a} f^R(d) \sqcap h(d)) = \bigsqcup_{c \neq a} (h(c) \sqcap -f^R(c)) \sqcup (f^R(c) \sqcap h(c)) = \bigsqcup_{c \neq a} h(c) \geqslant -h(a)$, as $\bigsqcup_{b \in S} h(b) = 1$.

On the other hand, let $\bigsqcup_{s\in M} R(s) \sqcap s(a) = p_3$. Then $p_2 = (h(a) \sqcap -f^R(a)) \sqcup (p_3 \sqcap h(a)) = (h(a) \sqcap -p_3 \sqcap q) \sqcup (p_3 \sqcap h(a)) = (h(a) \sqcap q) \sqcup (h(a) \sqcap p_3)$. Hence $p = p_1 \sqcup p_2 \geqslant -h(a) \sqcup (h(a) \sqcap q) \sqcup (h(a) \sqcap p_3) \geqslant q$.

□

To prove Atomicity we need some more lemmas.

Lemma D.4.1. Let $f \in M$. $\mathfrak{S}_S^B \models \forall v(E(v) \to \neg v \precnapprox f)$ just in case $\{f(a) \mid a \in S\}$ is an antichain in B.

Proof. Right to left direction. Let $f \in M$ be such that $\{f(a) \mid a \in S\}$ is an antichain. Fix some random $g \in M$. We will show that $[\![E(g) \to \neg g \precnapprox f]\!] = 1$. That is,

$$\bigsqcup_{a\in S} g(a) \leqslant (\bigsqcup_{b\in S} g(b) \sqcap -f(b)) \sqcup (\bigsqcap_{c\in S} g(c) \sqcup -f(c))$$

Fix some random $a \in S$. It is easy to see that

$$g(a) \sqcap -f(a) \leqslant \bigsqcup_{b\in S} g(b) \sqcap -f(b)$$
$$g(a) \sqcap (\bigsqcap_{c\in S\setminus\{a\}} -f(c)) \leqslant \bigsqcap_{c\in S} g(c) \sqcup -f(c)$$

Since $\{f(a) \mid a \in S\}$ is an antichain, $f(a) \leqslant (\bigsqcap_{c\in S\setminus\{a\}} -f(c))$. Hence, $g(a) \sqcap f(a) \leqslant \bigsqcap_{c\in S} g(c) \sqcup -f(c)$. Therefore,

$$g(a) = (g(a) \sqcap -f(a)) \sqcup (g(a) \sqcap f(a)) \leqslant (\bigsqcup_{b\in S} g(b) \sqcap -f(b)) \sqcup (\bigsqcap_{c\in S} g(c) \sqcup -f(c))$$

Left to right direction. Let $f \in M$ be such that for some $a, b \in S$, $f(a) \sqcap f(b) > 0$. Define $g \in M$ as follows: for any $c \in S$,

$$g(c) = \begin{cases} f(a) \sqcap -f(b) & \text{if } c = a; \\ f(c) & \text{if otherwise.} \end{cases}$$

It is easy to see that $[\![E(g)]\!] =]\!]E(f)]\!]$. And hence g is indeed in M. We will show that $[\![E(g) \to \neg g \precnapprox f]\!] < 1$. That is,

$$\bigsqcap_{a\in S} -g(a) \sqcup (\bigsqcup_{b\in S} g(b) \sqcap -f(b)) \sqcup (\bigsqcap_{c\in S} g(c) \sqcup -f(c)) < 1$$

Observe that $\bigsqcap_{a\in S} -g(a) = 0$, as $g \in M$. Also $\bigsqcup_{b\in S} g(b) \sqcap -f(b) = 0$. And $\bigsqcap_{c\in S} g(c) \sqcup -f(c) = g(a) \sqcup -f(a) = (f(a) \sqcap -f(b)) \sqcup -f(a) = -f(a) \sqcup -f(b) < 1$, as $f(a) \sqcap f(b) > 0$. Hence the whole thing is less than 1.

□

Lemma D.4.2. Let $f \in M$. $\mathfrak{S}^S_A \models At(f)$ just in case $\{f(a) \mid a \in S\}$ is a maximal antichain in B.

Proof. Recall that $At(f) = E(f) \wedge \forall v(E(v) \to \neg v \precnsim f)$. The result follows from the previous lemma as for any $f \in M$, $[\![E(f)]\!] = 1$.

□

Theorem D.5 (Atomicity). $\mathfrak{S}^B_S \models \forall v_1(E(v_1) \to \exists v_2(At(v_2) \wedge v_2 \precsim v_1))$.

Proof. Fix some random $f \in M$. Since $[\![E(f)]\!] = 1$, we need to show that $[\![\exists v_2(At(v_2) \wedge v_2 \precsim f)]\!] = 1$. Let $C = \{a \in S \mid f(a) \neq 0\}$. Enumerate C by $\alpha = |C|$: $C = \{a_1, \ldots, a_\beta, \ldots \mid \beta < \alpha\}$. Define $g \in M$ as follows: for any $c \in S$,

$$g(c) = \begin{cases} f(a_\beta) \sqcap (\prod\limits_{\gamma<\beta}) - f(a_\gamma) & \text{if } c = a_\beta \in C; \\ \\ f(c) = 0 & \text{if } c \notin C. \end{cases}$$

Hence $[\![E(g)]\!] = \bigsqcup_{a \in C} g(a) = \bigsqcup_{a \in C} f(a) = [\![E(f)]\!] = 1$. Also, $[\![g \precsim f]\!] = 1$. Since $\{g(a) \mid a \in S\}$ is an antichain, by Lemma D.4.1, $[\![\forall v(E(v) \to \neg v \precnsim g)]\!] = 1$. Hence $[\![At(g)]\!] = [\![E(f)]\!] = 1$.

□

Theorem D.6 (NoZero). $\mathfrak{S}^B_S \models \exists v_1 \exists v_2(v_1 \not\precsim v_2) \to \neg\exists v_3 \neg(E(v_3))$

Proof. This can be proven simply by showing that $[\![\neg\exists v_3 \neg(E(v_3))]\!] = 1$, as for any $f \in M$, $[\![E(f)]\!] = 1$.

□

Corollary D.6.1. $\mathfrak{S}^B_S$ is a model of ACM^-.

E The VE Models

In this section we prove the following result:

Theorem E.1. In any VE model, Transitivity, Supplementation, Fusion and Atomicity all have value 1, but NoZero has value 0.

Transitivity is proven in the same way as before.

Lemma E.1.1. For any $f \in N$, $[\![E(f)]\!] = \bigsqcup_{a \in S} f(a)$.

Proof. The same proof as in that of Lemma D.2.1.

□

Lemma E.1.2. For any $f_1, f_2 \in N$, $[\![f_1 \circ f_2]\!] = \bigsqcup_{a \in S} f_1(a) \sqcap f_2(a)$.

Proof. For this proof and many followings, we need to consider two cases. Case one is when $\bigsqcup_{a\in S} f_1(a) \sqcap f_2(a) = 0$. Then for any $a \in S$, $f_1(a) \sqcap f_2(a) = 0$. Then $[\![f_1 \circ f_2]\!] = \bigsqcup_{g\in N} \bigsqcup_{a\in S} g(a) \sqcap \bigsqcap_{b\in S} g(b) \Rightarrow (f_1(b) \sqcap f_2(b)) = \bigsqcup_{g\in N} \bigsqcup_{a\in S} g(a) \sqcap \bigsqcap_{b\in S} -g(b) = 0$.

Case two is when $\bigsqcup_{a\in S} f_1(a) \sqcap f_2(a) > 0$. Then define $f \in N$ such that for any $a \in S$, $f(a) = f_1(a) \sqcap f_2(a)$. It is easy to see that $[\![f \precsim f_1]\!] = [\![f \precsim f_2]\!] = 1$.

$[\![f_1 \circ f_2]\!] = [\![\exists v(E(v) \wedge v \precsim f_1 \wedge v \precsim f_2)]\!] = \bigsqcup_{g\in S^B} [\![E(g) \wedge g \precsim f_1 \wedge g \precsim f_2]\!]$.

Fix some random $g \in S^B$, $[\![E(g) \wedge g \precsim f_1 \wedge g \precsim f_2]\!] = \bigsqcup_{a\in S} g(a) \sqcap \bigsqcap_{b\in S} (g(b) \Rightarrow (f_1(b) \sqcap f_2(b))) \leqslant \bigsqcup_{a\in S} g(a) \sqcap (g(a) \Rightarrow (f_1(a) \sqcap f_2(a))) \leqslant \bigsqcup_{a\in S} f_1(a) \sqcap f_2(a) = [\![E(f)]\!] = [\![E(f) \wedge f \precsim f_1 \wedge f \precsim f_2]\!]$. Hence $\bigsqcup_{g\in S^B} [\![E(g) \wedge g \precsim f_1 \wedge g \precsim f_2]\!] = [\![E(f) \wedge f \precsim f_1 \wedge f \precsim f_2]\!] = [\![E(f)]\!] = \bigsqcup_{a\in S} f_1(a) \sqcap f_2(a)$.

□

Theorem E.2 (Supplementation)**.** $\mathfrak{S}_V^B \models \forall v_1 \forall v_2 (v_2 \not\precsim v_1 \rightarrow \exists v_3(E(v_3) \wedge v_3 \precsim v_2 \wedge \neg v_1 \circ v_3))$.

Proof. Let $f_1, f_2 \in N$. We want to show that $[\![f_2 \not\precsim f_1]\!] \leqslant [\![\exists v(E(v) \wedge v \precsim f_2 \wedge \neg f_1 \circ v)]\!]$. $[\![f_2 \not\precsim f_1]\!] = \bigsqcup_{a\in S} -f_1(a) \sqcap f_2(a)$.

Again, there are two cases. If $[\![f_2 \not\precsim f_1]\!] = 0$, then we are done. If $[\![f_2 \not\precsim f_1]\!] > 0$, then define $f \in N$ such that for any $a \in S$, $f(a) = -f_1(a) \sqcap f_2(a)$. We can easily show that $[\![f \precsim f_2]\!] = 1$. Also, $[\![E(f)]\!] = \bigsqcup_{a\in S} -f_1(a) \sqcap f_2(a)$, by Lemma E.1.1, and $[\![\neg f_1 \circ f]\!] = -(\bigsqcup_{a\in S} f_1(a) \sqcap -(f_1(a) \sqcap f_2(a))) = 1$, by Lemma E.1.2. Hence $[\![\exists v(E(v) \precsim f_2 \wedge \neg f_1 \circ v)]\!] \geqslant [\![E(f) \wedge f \precsim f_2 \wedge \neg f_1 \circ f]\!] = \bigsqcup_{a\in S} -f_1(a) \sqcap f_2(a) = [\![f_2 \not\precsim f_1]\!]$.

□

Theorem E.3 (Fusion)**.** $\mathfrak{S}_V^B \models \forall X_1(\exists v_1 X_1(v_1) \rightarrow \exists v_2(FU(v_2, X_1))$.

Proof. Let $R \in D_M^1$. Again, there are two cases. Case one: $\bigsqcup_{a\in S} \bigsqcup_{g\in N} R(g) \sqcap g(a) = 0$. Then for any $g \in N, a \in S$, $R(g) \sqcap g(a) = 0$. This case can be proven easily by unpacking the definitions. Case two: $\bigsqcup_{a\in S} \bigsqcup_{g\in N} R(g) \sqcap g(a) > 0$. Then define $f^R \in N$: for any $a \in S$, let $f(a) = \bigsqcup_{g\in S^B} R(g) \sqcap g(a)$. We will show that $[\![FU(f^R, R)]\!] = [\![\forall v_2(R(v_2) \rightarrow v_2 \precsim f^R) \wedge \forall v_3(v_3 \precsim f^R \wedge E(v_3) \rightarrow \exists v_4(R(v_4) \wedge v_3 \circ v_4))]\!] = 1$.

$[\![\forall v_2(R(v_2) \rightarrow v_2 \precsim f^R)]\!] = \bigsqcap_{h\in S^B} R(h) \Rightarrow (\bigsqcap_{a\in S} h(a) \Rightarrow f^R(a))$. Fix some $h \in N$. Then $-R(h) \sqcup (\bigsqcap_{a\in S} -h(a) \sqcup (\bigsqcup_{g\in S^B} R(g) \sqcap g(a))) = \bigsqcap_{a\in S} -(R(h) \sqcap h(a)) \sqcup \bigsqcup_{g\in S^B} R(g) \sqcap g(a)) \geqslant \bigsqcap_{a\in S} -(R(h) \sqcap h(a)) \sqcup (R(h) \sqcap g(h)) = 1$.

$[\![\forall v_3(v_3 \precsim f^R \wedge E(v_3) \rightarrow \exists v_4(R(v_4) \wedge v_3 \circ v_4))]\!] = \bigsqcap_{g\in S^B}([\![g \precsim f^R]\!] \sqcap [\![E(g)]\!]) \Rightarrow (\bigsqcup_{h\in S^B}(R(h) \sqcap [\![h \circ g]\!]))$. Fix some $g \in N$. $\bigsqcup_{h\in S^B}(R(h) \sqcap [\![h \circ g]\!]) = \bigsqcup_{h\in S^B} R(h) \sqcap$

$\bigsqcup_{a\in S} h(a) \sqcap g(a) = \bigsqcup_{a\in S} \bigsqcup_{h\in S^B} R(h) \sqcap h(a) \sqcap g(a) = \bigsqcup_{a\in S} f^R(a) \sqcap g(a) = [\![f^R \circ g]\!]$. But $[\![f^R \circ g]\!] = [\![\exists v_1(E(v_1) \wedge v_1 \precsim f^R \wedge v_1 \precsim g)]\!] = \bigsqcup_{t\in S^B} [\![E(t)]\!] \sqcap [\![t \precsim f^R]\!] \sqcap [\![t \precsim g]\!] \geqslant [\![E(g)]\!] \sqcap [\![g \precsim f^R]\!]$.

□

Lemma E.3.1. Let $f \in N$. $\mathfrak{S}_V^S \models At(f)$ just in case $\{f(a) \mid a \in S\}$ is a maximal antichain in B.

Proof. Using the same proof as in Lemma D.4.1 we can show that for any $f \in N$, $\mathfrak{S}_V^B \models \forall v(E(v) \to \neg v \precnsim f)$ just in case $\{f(a) \mid a \in S\}$ is an antichain in B.

Recall that $At(f) = E(f) \wedge \forall v(E(v) \to \neg v \precnsim f)$. Suppose $[\![At(f)]\!] = 1$. Then $\{f(a) \mid a \in S\}$ is an antichain. Also, since $\bigsqcup_{a\in S} f(a) = [\![E(f)]\!] = 1$, $\{f(a) \mid a \in S\}$ is a maximal antichain. Similarly, suppose $\{f(a) \mid a \in S\}$ is a maximal antichain, then $[\![E(f)]\!] = \bigsqcup_{a\in S} f(a) = 1$. Also, $[\![\forall v(E(v) \to \neg v \precnsim f)]\!] = 1$. Hence $[\![At(f)]\!] = 1$.

□

Theorem E.4 (Atomicity)**.** $\mathfrak{S}_V^B \models \forall v_1(E(v_1) \to \exists v_2(At(v_2) \wedge v_2 \precsim v_1))$.

Proof. The same proof as in Theorem D.5, using the previous lemma.

□

Theorem E.5 (NoZero is false.)**.** $\mathfrak{S}_V^B \models \neg(\exists v_1 \exists v_2(v_1 \not\precsim v_2) \to \neg \exists v_3 \neg(E(v_3)))$

Proof. This can be proven by showing two things. First, $[\![\exists v_1 \exists v_2(v_1 \not\precsim v_2)]\!]$ has value 1. $[\![\exists v_1 \exists v_2(v_1 \not\precsim v_2)]\!] = \bigsqcup_{f_1, f_2 \in N} \bigsqcup_{a\in S} f_1(a) \sqcap -f_2(a)$. Let $f_1, f_2 \in N$ be such that for some $a \in S$, $f_1(a) = 1$ and $f_2(a) = 0$. Then $f_1(a) \sqcap -f_2(a) = 1$. Second, $[\![\neg \exists v_3 \neg(E(v_3))]\!]$ has value 0. Define $f^p \in S^B$ to be the constant function that takes every $a \in S$ to p, where $0 < p < 1$, and $f^{-p} \in N$ to be the constant function that takes every $a \in S$ to $-p$. Then $[\![E(f^p)]\!] = p$ and $[\![E(f^{-p})]\!] = -p$. Hence $[\![\neg \exists v_3 \neg(E(v_3))]\!] \geqslant [\![E(f^p)]\!] \sqcap [\![E(f^{-p})]\!] = 0$.

□

Corollary E.5.1. $\mathfrak{S}_V^B$ is a model of *MACM*, but not a model of ACM^-.

F Identity and Anti-Symmetry

Recall that an atomic Boolean model is a VI model if it is $SEVI$ or $VEVI$, and similarly is a TI model if it is $SETI$ or $VETI$.

Proposition F.1. In any VI model, Anti-Symmetry has value 1.

Proof. Directly follows from Vague-Identity: for any $f_1, f_2 \in M/N$, $[\![f_1 = f_2]\!] = \bigsqcap_{a \in S} f_1(a) \Leftrightarrow f_2(a) = [\![f_1 \precsim f_2]\!] \sqcap [\![f_2 \precsim f_1]\!]$.

□

Proposition F.2. In any TI model, Anti-Symmetry has value 0.

Proof. Define $f_1 : S \to B$ as follows: for some $a \in S$, $f_1(a) = p$, where $0 < p < 1$; for any $b \neq a \in S$, $f_1(b) = 1$. Define $f_2 : S \to B$ as follows: $f_2(a) = -p$ and for any $b \neq a \in S$, $f_2(b) = 1$. Define $f : S \to B$ as follows: for any $c \in S$, $f(c) = 1$. It is easy to see that $f_1, f_2 \in M \subseteq N$.

It is also easy to see that $[\![f_1 \precsim f]\!] = [\![f_2 \precsim f]\!] = 1$. And $[\![f \precsim f_1]\!] = 1 \Rightarrow p = p$, $[\![f \precsim f_2]\!] = 1 \Rightarrow -p = -p$. Also, since f, f_1, f_2 are different functions, $[\![f_1 = f]\!] = [\![f_1 = f]\!] = 0$.

Hence $[\![f_1 \precsim f \wedge f \precsim f_1 \to f = f_1]\!] = (1 \sqcap p) \Rightarrow 0 = -p$. And $[\![f_2 \precsim f \wedge f \precsim f_2 \to f = f_2]\!] = (1 \sqcap -p) \Rightarrow 0 = p$. Hence $[\![\forall v_1 \forall v_2 (v_1 \precsim v_2 \wedge v_2 \precsim v_1 \to v_1 = v_2)]\!] \leqslant p \sqcap -p = 0$.

□

Corollary F.0.1. In any $SEVI$ model, Transitivity, Supplementation, Fusion, Atomicity, NoZero and Anti-Symmetry all have value 1.

Corollary F.0.2. In any $SETI$ model, Transitivity, Supplementation, Fusion, Atomicity and NoZero all have value 1, but Anti-Symmetry has value 0.

Corollary F.0.3. In any $VEVI$ model, Transitivity, Supplementation, Fusion, Atomicity, and Anti-Symmetry all have value 1, but NoZero has value 0.

Corollary F.0.4. In any $VETI$ model, Transitivity, Supplementation, Fusion and Atomicity all have value 1, but NoZero and Anti-Symmetry have value 0.