BOOLEAN-VALUED MODELS OF SET THEORY WITH URELEMENTS

XINHE WU, BOKAI YAO

ABSTRACT. We study Boolean-valued models of set theory with a proper class of urelements. We prove the fundamental theorem for Boolean-valued models with urelements concerning axiom preservation over $\mathrm{ZFCU_R}$. We then show that certain axioms such as DC_{ω_1}-scheme are preserved only by certain complete Boolean algebras. We then turn to the property of fullness. Since the standard Boolean-valued models with urelements are almost never full, we provide a different construction $\overline{U^{\mathbb{B}}}$. The standard construction is shown to be an elementary substructure of $\overline{U^{\mathbb{B}}}$. And we prove that over $\mathrm{ZFCU_R}$, the Axiom of Collection is equivalent to the principle that for every complete Boolean algebra $\mathbb{B}$, $\overline{U^{\mathbb{B}}}$ is full.

1. Introduction

A Boolean-valued model $M^{\mathbb{B}}$ for a first-order language $\mathcal{L}$ consists of a domain of $\mathbb{B}$-names together with a B-valued truth assignment $[\![\ ]\!]$, which assigns a $\mathbb{B}$-value to each assertion in $\mathcal{L}$ about the $\mathbb{B}$-names in a way that obeys the axioms of first-order logic. Boolean-valued models of set theory, as an elegant presentation of the method of forcing, have been widely studied (e.g., see [3], [9], and [7]). In this paper, we study Boolean-valued models of set theory with *urelements*. As the existing studies of this topic (e.g., [4], [5] and [6]) assume that all urelements form a set, this motivates studying Boolean-valued models of set theory that allows a proper class of urelements. In Section 2, we start with a standard way of constructing a Boolean-valued universe $U^{\mathbb{B}}$ within a weak urelement set theory $\mathrm{ZFCU_R}$. Since there is known to be a hierarchy of axioms on the basis of $\mathrm{ZFCU_R}$, it is natural to investigate which axioms will be preserved in $U^{\mathbb{B}}$ given a complete Boolean algebra $\mathbb{B}$. We prove a fundamental theorem (Theorem 2.10) concerning axiom preservation in $U^{\mathbb{B}}$ over $\mathrm{ZFCU_R}$ which clarifies the issue. We also show that $U^{\mathbb{B}}$ can both destroy and recover certain natural axioms. In Section 3, we turn to the property of *fullness*. Since, unlike the situation in ZFC, with urelements $U^{\mathbb{B}}$ is almost never full, we construct a different Boolean-valued universe, $\overline{U^{\mathbb{B}}}$, which is proved to be an elementary extension of $U^{\mathbb{B}}$ (Theorem 3.11). And we show that over $\mathrm{ZFCU_R}$, $\overline{U^{\mathbb{B}}}$ is full for every complete Boolean algebra $\mathbb{B}$ just in case the Axiom of Collection holds (Theorem 3.12).

The rest of this section reviews some basic facts and known results about ZFC with urelements. The first-order language of set theory with urelements $\{\in, \mathscr{A}\}$ contains an additional uniary predicate $\mathscr{A}$ for urelements. It is always an axiom that urelements have no members, and we allow a proper class of urelements. The standard axioms of ZFC will be modified to allow urelements, e.g., the Axiom of Extensionality in this context will assert that *sets* with the same members are identical. The Axiom of Collection is the scheme that for any w and u, if $\forall x \in w \exists y \varphi(x, y, u)$, then there is a z such that $\forall x \in w \exists y \in z \varphi(x, y, u)$.

Definition 1.1. ZCU is the urelement set theory which includes the following axioms: Extensionality, Foundation, Pairing, Union, Powerset, Infinity, Separation and the Axiom of Choice.
$\mathrm{ZFCU_R}$ = ZCU + Replacement.
ZFCU = ZCU + Collection.

Definition 1.2 ($\mathrm{ZFCU_R}$)**.** For every set of urelements A, the *tail cardinal* of A, if exists, is the largest cardinal κ such that there is a set of urelements B of size κ that is disjoint from A.

We shall also consider the following axioms which have appeared in the literature (possibly except for Tail).

(DC_ω-scheme) If for every x there is a y such that $\varphi(x, y, u)$, then there is an ω-sequence $\langle x_n : n < \omega \rangle$ such that $\varphi(x_n, x_{n+1}, u)$ for every n.

(Plenitude) For any cardinal κ, there is a set of urelements of size κ.

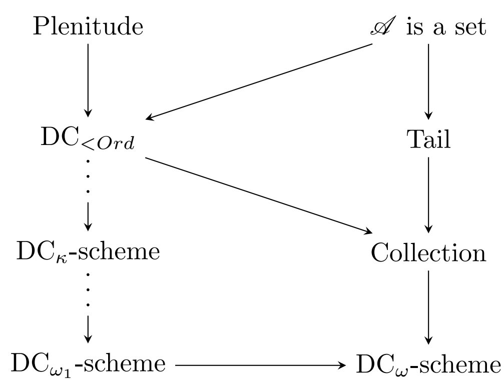
(Tail) For every set of urelements has a tail cardinal.

DC_ω-scheme is a class version of the Axiom of Dependent Choice. For any infinite κ, we can formulate DC_κ-scheme as follows.

(DC_κ-scheme) If for every x there is a y such that $\varphi(x, y, u)$, then there is a function f on κ such that such that $\varphi(f \restriction \alpha, f(\alpha), u)$ for every $\alpha < \kappa$.

$\mathrm{DC}_{<Ord}$ holds if DC_κ-scheme holds for all κ. When $\mathscr{A}$ is a set, both Collection and $\mathrm{DC}_{<Ord}$ follow from $\mathrm{ZFCU_R}$. The following two theorems are proved in the second author's unpublished work [14].

Theorem 1.1 ([14])**.** Over $\mathrm{ZFCU_R}$, the following implication diagram holds. The diagram is also complete: if the diagram does not indicate φ implies ψ, then $\mathrm{ZFCU_R}$ + $\varphi \nvdash \psi$. [1] □



[1]The fact that Collection implies DC_ω-scheme is first proved in [1].

Theorem 1.2 ([14])**.** Let κ be an infinite cardinal. Over $\mathrm{ZFCU_R}$, if the tail cardinal of every set of urelements is at least κ , then DC_κ-scheme holds. □

In particular, $\mathrm{ZFCU_R}$ cannot prove DC_ω-scheme and is thus strictly weaker than ZFCU. For example, it is folklore that there are models of $\mathrm{ZFCU_R}$ where the urelements form a proper class but every set of urelements is finite. In a model as such, every set can be properly extended by another set of urelements, but no infinite sequence of increasing sets of urelements can exist, so DC_ω-scheme fails. Collection fails in such model too, as for every $n < \omega$ there is a set of urelements of size n, but we cannot collection them into a set. Moreover, many ZFC-theorems, such as the reflection principle, are not provable in $\mathrm{ZFCU_R}$ but in ZFCU (see [14] and [8]). For this reason, one might consider $\mathrm{ZFCU_R}$ as an inadequate urelement set theory. However, as $\mathrm{ZFCU_R}$ still proves transfinite recursion and thus suffices for the basic construction of a Boolean-valued universe, it is natural to work in this weak theory in order to obtain stronger results. And as we shall see, a Boolean-valued universe built inside some universe of $\mathrm{ZFCU_R}$ without Collection can in fact satisfy the stronger theory ZFCU.

Notation and basic facts. The symbol $\mathscr{A}$ will also stand for the class of all urelements; $x \subseteq \mathscr{A}$ abbreviates "x is a set of urelements"; U stands for the class $\{x : x = x\}$; the lowercase letters $a, b, c, \ldots$ denote urelements and the uppercase letters $A, B, C, \ldots$ stand for sets of urelements. $x \sim y$ abbreviates "x is equinumerous with y" and $x \preceq y$ abbreviates "there is an injection from x to y". For every x, the kernel of x, $ker(x)$, is the set of urelements in the transitive closure of $\{x\}$. For any set A of urelements, the class $V(A)$ is the accumulative hierarchy built from A by iterating the powerset operation. Namely,

$V_0(A) = A$;
$V_{\alpha+1}(A) = P(V_\alpha(A)) \cup V_\alpha(A)$;
$V_\gamma(A) = \bigcup_{\alpha<\gamma} V_\alpha(A)$, where γ is a limit ordinal;
$V(A) = \bigcup_{\alpha \in Ord} V_\alpha(A)$.

For every x, $ker(x) \subseteq A$ iff $x \in V(A)$. A very important feature of U is that it adimits non-trivial automorphisms: for any definable permutation i of $\mathscr{A}$, i can be extended to an automorphism of U by letting $ix = \{iy : y \in x\}$ for every x; moreover, $i(x) = x$ whenever i point-wise fixes $ker(x)$ (i.e., for all $a \in ker(x), i(a) = a$).

2. $U^{\mathbb{B}}$

2.1. **The basic construction.** Recall that in ZFC, given a complete Boolean algebra $\mathbb{B} \in V$, by transfinite recursion we can define a $\mathbb{B}$ name to be a function from a set of $\mathbb{B}$-names to $\mathbb{B}$. The Boolean-valued universe $V^{\mathbb{B}}$ is then the class of all $\mathbb{B}$-names, which is a definable class inside V. In particular, $\emptyset$ will be its own $\mathbb{B}$-name. A straightforward generalization of this construction in U is to let each urelement be its own $\mathbb{B}$-name. This motivates the following definition which is considered in [4].

Definition 2.1 ($\mathrm{ZFCU_R}$)**.** Let $\mathbb{B} \in U$ be a complete Boolean algebra.

(1) τ is a $\mathbb{B}$-name iff τ is a urelement, or τ is a function from a set of $\mathbb{B}$-names to $\mathbb{B}$.
(2) $U^{\mathbb{B}} = \{\tau \in U : \tau \text{ is a } \mathbb{B}\text{-name}\}$

(3) $\mathcal{L}_{\mathbb{B}}$ is the extended language of urelement set theory containing each $\mathbb{B}$-name as a constant symbol. $\mathcal{AL}_{\mathbb{B}}$ is class of all atomic formulas in $\mathcal{L}_{\mathbb{B}}$.
(4) The Boolean evaluation function $[\![\ \ ]\!] : \mathcal{AL}_{\mathbb{B}} \to \mathbb{B}$ is defined as follows by recusion.

$$[\![\tau \subseteq \sigma]\!] = \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\eta \in \sigma]\!]$$

$$[\![\tau = \sigma]\!] = \begin{cases} 1 & \text{if } \tau, \sigma \in \mathscr{A} \text{ and } \tau = \sigma \\ 0 & \text{if } \tau \in \mathscr{A} \text{ or } \sigma \in \mathscr{A}, \text{ and } \tau \neq \sigma \\ [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!] & \text{if } \tau, \sigma \notin \mathscr{A} \end{cases}$$

$$[\![\mathscr{A}(\tau)]\!] = \begin{cases} 1 & \text{if } \tau \in \mathscr{A} \\ 0 & \text{if } \tau \notin \mathscr{A} \end{cases}$$

The evaluation function $[\![\ \ ]\!]$ can be extended into all formulas in $\mathcal{L}_{\mathbb{B}}$ in the standard way. Namely,

$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$;
$[\![\neg\varphi]\!] = \neg[\![\varphi]\!]$;
$[\![\exists x \varphi]\!] = \bigvee_{\tau \in U^{\mathbb{B}}} [\![\varphi(\tau)]\!]$.

$U^{\mathbb{B}}$ will also denote the Boolean-valued structure $\langle U^{\mathbb{B}}, [\![\ ]\!] \rangle$, where $U^{\mathbb{B}} \models \varphi$ means $[\![\varphi]\!] = 1$. With some trivial modifications of the proofs in [3, p. 24-26], one can show that all the axioms of the first-order predicate calculus have value 1 in $U^{\mathbb{B}}$, and all of the rules of inferences are valid in $U^{\mathbb{B}}$. The following facts will be frequently used, and we refer the reader to [3, p.27-47] for their proofs.

Proposition 2.1. For any formula $\varphi(x)$ and any τ in $U^{\mathbb{B}}$,

(1) $[\![\exists x \in \tau(\varphi(x))]\!] = \bigvee_{\eta \in dom(\tau)} (\tau(\eta) \wedge [\![\varphi(\eta)]\!])$.
(2) $[\![Ord(\tau)]\!] = \bigvee_{\alpha \in ORD} [\![\tau = \check{\alpha}]\!]$.
(3) $[\![\exists x(Ord(x) \wedge \varphi(x))]\!] = \bigvee_{\alpha \in ORD} \varphi(\check{\alpha})$.
(4) (The Induction Inprinciple) $\forall \tau \in U^{\mathbb{B}}(\forall \eta \in dom(\tau)\varphi(\eta) \to \varphi(\tau)) \to \forall \tau \in U^{\mathbb{B}}(\varphi(\tau))$. □

There is a canonical way of representing U in $U^{\mathbb{B}}$: for any urelement a, we let $\check{a} = a$; for any set x, $\check{x} = \{\langle \check{y}, 1 \rangle \mid y \in x\}$. It is routine to check that $x \mapsto \check{x}$ preserves Δ_0 assertions: $U \models \varphi(x_0, ..., x_n)$ iff $U^{\mathbb{B}} \models \varphi(\check{x_0}, ..., \check{x_n})$ for any Δ_0 formula $\varphi \in \mathcal{L}_{\mathbb{B}}$. Furthermore, by the same arguments as in [3, p.37-45], we can show that all the axioms of ZCU have vaule 1 in $U^{\mathbb{B}}$.

Theorem 2.2. $U^{\mathbb{B}} \models$ ZCU. □

For every $\tau \in U^{\mathbb{B}}$, let $A_\tau = \{a \in \mathscr{A} \mid a \in dom(\tau)\}$. $U^{\mathbb{B}}$ thinks that every set of urelement τ is covered by the set urelements $\check{A}_\tau$.

Lemma 2.3. $[\![\tau \subseteq \mathscr{A}]\!] = [\![\tau \subseteq \check{A}_\tau]\!]$.

Proof. $[\![\tau \subseteq \mathscr{A}]\!] = \bigwedge_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![\mathscr{A}(\eta)]\!]) = \bigwedge_{\eta \in dom(\tau) \setminus \mathscr{A}} \neg\tau(\eta) = [\![\tau \subseteq \check{A}_\tau]\!]$. □

2.2. **The fundamental theorem of $U^{\mathbb{B}}$.** We now turn to the fundamental theorem of $U^{\mathbb{B}}$ concerning axiom preservation. For now on, it is always assumed that $\mathrm{ZFCU_R}$ holds in U and $\mathbb{B}$ is a complete Boolean algebra in U.

Lemma 2.4. If $U \models$ Collection, then $U^{\mathbb{B}} \models$ Collection.

Proof. It suffices to show that for every $\tau \in U^{\mathbb{B}}$, there is a $\rho \in U^{\mathbb{B}}$ such that [2]

$$[\![\forall x \in \tau \exists y \varphi(x,y)]\!] \leqslant [\![\forall x \in \tau \exists y \in \rho \varphi(x,y)]\!].$$

Now fix $\tau \in U^{\mathbb{B}}$. For any $\sigma \in dom(\tau)$, let $X_\sigma = \{p \in \mathbb{B} \mid \exists \pi \in U^{\mathbb{B}}(p = [\![\varphi(\sigma,\pi)]\!])\}$. By Collection and Separation in U, it follows that there is a $Y_\sigma \subseteq U^{\mathbb{B}}$ such that $\forall p \in X_\sigma \exists \pi \in Y_\sigma (p = [\![\varphi(\sigma,\pi)]\!])$. Then $[\![\exists y \varphi(\sigma,x)]\!] = \bigvee_{\pi \in Y_\sigma} [\![\varphi(\sigma,\pi)]\!]$. This shows that for every $\sigma \in dom(\tau)$, there is a $Y_\sigma \subseteq U^{\mathbb{B}}$ such that $[\![\exists y \varphi(\sigma,x)]\!] = \bigvee_{\pi \in Y_\sigma} [\![\varphi(\sigma,\pi)]\!]$. By Collection again, we can collect those Y_σ into a set $\bar{Y}$. Now let ρ be $((\bigcup \bar{Y}) \cap U^{\mathbb{B}}) \times \{1\}$. For any $\sigma \in dom(\tau)$, $[\![\exists y \varphi(\sigma,x)]\!] = \bigvee_{\pi \in \bigcup \bar{Y}} [\![\varphi(\sigma,\pi)]\!] = [\![\exists y \in \rho\, \varphi(x,y)]\!]$. Thus, ρ is as desired. □

The case with Replacement in $U^{\mathbb{B}}$ is trickier. The standard proof of Replacement having value 1 in $V^{\mathbb{B}}$ uses Collection in V, which does not work for our purpose since Collection is not provable in $\mathrm{ZFCU_R}$. Our proof will use the idea of purification, which can be found in [4]. Given a set of urelements A and a $\mathbb{B}$-name τ, we can construct the A-purification of τ, $\overset{A}{\tau}$, which is also a $\mathbb{B}$-name. Intuitively, $\overset{A}{\tau}$ is what we get by "purifying off" the urelements appeared in the construction of τ that are not in A. We wish to show that τ will become similar to $\overset{A}{\tau}$ from the perspective of $U^{\mathbb{B}}$ when A contains enough urelements.

Definition 2.2 (Purification)**.** Let A be a set of urelements. For any urelement $a \in \mathscr{A}$, we define $\overset{A}{a}$ as a. Let τ be a $\mathbb{B}$-name. We define $\overset{A}{\tau} \in U^{\mathbb{B}}$ recursively as follows:

$$dom(\overset{A}{\tau}) = \{\overset{A}{\eta} \mid \eta \in dom(\tau) \cap A\}$$

Let $\mu \in dom(\overset{A}{\tau})$. We define $X_\mu = \{\eta \in dom(\tau) \mid \overset{A}{\eta} = \mu\}$, and

$$\overset{A}{\tau}(\mu) = \bigvee_{\eta \in X_\mu} \tau(\eta)$$

Definition 2.3. Let a, b be two urelements. $i^a_b : U \to U$ is the automorphism generated by the definable permutation of $\mathscr{A}$ which only swaps a and b.

Note that if μ is a $\mathbb{B}$-name and c is a urelement with $c \notin ker(\mu)$. Then for any urelement b, $i^b_c(\mu)$ is a $\mathbb{B}$-name such that $dom(i^b_c(\mu)) = \{i^b_c(\eta) \mid \eta \in dom(\mu)\}$; and for any $i^b_c(\eta) \in dom(i^b_c(\mu))$, $i^b_c(\mu)(i^b_c(\eta)) = \mu(\eta)$.

Lemma 2.5. Let c be a urelement. Let η, μ be $\mathbb{B}$-names such that $c \notin ker(\eta) \cup ker(\mu)$. Then, for any urelement b,

$$[\![\eta = i^b_c(\mu)]\!] \leqslant [\![\eta = \mu]\!]$$

[2] For readability, we shall omit parameters when doing so does not undermine the general idea of the proof.

Proof. We use the induction principle on μ. Since $c \notin ker(\eta) \cup ker(\mu)$, for any $\nu \in dom(\mu), \gamma \in dom(\eta)$, $c \notin ker(\nu) \cup ker(\gamma)$. By inductive hypothesis, then, for any $\nu \in dom(\mu), \gamma \in dom(\eta)$, any urelement b,

$$[\![\gamma = i_c^b(\nu)]\!] \leqslant [\![\gamma = \nu]\!]$$

We first show that $[\![\eta \subseteq i_c^b(\mu)]\!] \leqslant [\![\eta \subseteq \mu]\!]$. That is,

$$\bigwedge_{\gamma \in dom(\eta)} \eta(\gamma) \Rightarrow [\![\gamma \in i_c^b(\mu)]\!] \leqslant \bigwedge_{\gamma \in dom(\eta)} \eta(\gamma) \Rightarrow [\![\gamma \in \mu]\!]$$

It suffices to show that for any $\gamma \in dom(\eta)$, $[\![\gamma \in i_c^b(\mu)]\!] \leqslant [\![\gamma \in \mu]\!]$. Since $[\![\gamma \in i_c^b(\mu)]\!] = \bigvee_{\nu \in dom(\eta)} [\![\gamma = i_c^b(\nu)]\!] \wedge \mu(\nu)$, $[\![\gamma \in i_c^b(\mu)]\!] \leqslant [\![\gamma \in \mu]\!]$ by inductive hypothesis. By a similar reasoning we also have $[\![i_c^b(\mu) \subseteq \eta]\!] \leqslant [\![\mu \subseteq \eta]\!]$. □

Lemma 2.6. Let A be a set of urelements, τ be a $\mathbb{B}$-name. Then,

$$[\![\overset{A}{\tau} \subseteq \tau]\!] = \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]$$

Similarly, for any urelement $c \notin ker(\tau)$, any urelement b,

$$[\![i_c^b(\tau) \subseteq \tau]\!] = \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![i_c^b(\eta) \in \tau]\!]$$

Proof. We need to show that

$$\bigwedge_{\mu \in dom(\overset{A}{\tau})} \overset{A}{\tau}(\mu) \Rightarrow [\![\mu \in \tau]\!] = \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]$$

Recall that for each $\mu \in dom(\overset{A}{\tau})$, we let $X_\mu = \{\eta \in dom(\tau) \mid \overset{A}{\eta} = \mu\}$. Hence the above equation holds because for any $\mu \in dom(\overset{A}{\tau})$,

$$\begin{aligned} \overset{A}{\tau}(\mu) \Rightarrow [\![\mu \in \tau]\!] &= \bigwedge_{\eta \in X_\mu} \tau(\eta) \Rightarrow [\![\mu \in \tau]\!] \\ &= \bigwedge_{\eta \in X_\mu} \tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!] \end{aligned}$$

The second statement holds for similar reasons. □

Lemma 2.7. Let A be a set of urelements, τ be a $\mathbb{B}$-name. Then, for any $\mathbb{B}$-name σ,

$$[\![\sigma \in \overset{A}{\tau}]\!] = \bigvee_{\eta \in dom(\tau)} \tau(\eta) \wedge [\![\sigma = \overset{A}{\eta}]\!]$$

Similarly, for any urelement $c \notin ker(\tau)$, any urelement b,

$$[\![\sigma \in i_c^b(\tau)]\!] = \bigvee_{\eta \in dom(\tau)} \tau(\eta) \wedge [\![\sigma = i_c^b(\eta)]\!]$$

Proof. We need to show that

$$\bigvee_{\mu \in dom(\overset{A}{\tau})} \overset{A}{\tau}(\mu) \wedge [\![\sigma = \mu]\!] = \bigvee_{\eta \in dom(\tau)} \tau(\eta) \wedge [\![\sigma = \overset{A}{\eta}]\!]$$

It holds because for any $\mu \in dom(\overset{A}{\tau})$, $\overset{A}{\tau}(\mu) \wedge [\![\sigma = \mu]\!] = \bigvee_{\eta \in X_\mu} \tau(\eta) \wedge [\![\sigma = \overset{A}{\eta}]\!]$. The second statement holds for similar reasons.

□

Lemma 2.8. Let A be a set of urelements, τ be a $\mathbb{B}$-name and c be a urelement such that $c \notin ker(\tau) \cup A$. Then,

$$\bigwedge_{b \in (ker(\tau) \setminus A)} [\![\tau = i^b_c(\tau)]\!] \leqslant [\![\tau = \overset{A}{\tau}]\!]$$

Proof. For any $\mathbb{B}$-name τ, let C_τ be $ker(\tau) \setminus A$. We use the induction principle on τ. Since $c \notin ker(\tau) \cup A$, for any $\eta \in dom(\tau)$, $c \notin ker(\eta) \cup A$. Assume as inductive hypothesis that for any $\eta \in dom(\tau)$,

$$\bigwedge_{b \in C_\eta} [\![\eta = i^b_c(\eta)]\!] \leqslant [\![\eta = \overset{A}{\eta}]\!]$$

Our goal is to show that

$$\bigwedge_{b \in C_\tau} [\![\tau \subseteq i^b_c(\tau)]\!] \wedge [\![i^b_c(\tau) \subseteq \tau]\!] \leqslant [\![\tau \subseteq \overset{A}{\tau}]\!] \wedge [\![\overset{A}{\tau} \subseteq \tau]\!] \tag{1}$$

Observe that

$$\begin{aligned}
[\![\tau \subseteq \overset{A}{\tau}]\!] \wedge [\![\overset{A}{\tau} \subseteq \tau]\!] &= \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow ([\![\eta \in \overset{A}{\tau}]\!] \wedge [\![\overset{A}{\eta} \in \tau]\!]) \\
\text{(By 2.7)} \quad &= \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow (\bigvee_{\mu \in dom(\tau)} \tau(\mu) \wedge [\![\eta = \overset{A}{\mu}]\!] \wedge [\![\overset{A}{\eta} = \mu]\!]) \\
&\geqslant \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow (\tau(\eta) \wedge [\![\eta = \overset{A}{\eta}]\!] \wedge [\![\overset{A}{\eta} = \eta]\!]) \\
&= \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\eta = \overset{A}{\eta}]\!] \\
\text{(By IH)} \quad &\geqslant \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow \bigwedge_{b \in C_\eta} [\![\eta = i^b_c(\eta)]\!] \\
&= \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow \bigwedge_{b \in C_\tau} [\![\eta = i^b_c(\eta)]\!]
\end{aligned}$$

where the first line holds by 2.6 and the last line holds because for any $b \in C_\tau \setminus C_\eta$, $i^b_c(\eta) = \eta$.

Also observe that

$$\begin{aligned}
\bigwedge_{b \in C_\tau} [\![\tau \subseteq i^b_c(\tau)]\!] \wedge [\![i^b_c(\tau) \subseteq \tau]\!] &= \bigwedge_{b \in C_\tau} \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\eta \in i^b_c(\tau)]\!] \wedge [\![i^b_c(\eta) \in \tau]\!] \\
&= \bigwedge_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow \bigwedge_{b \in C_\tau} [\![\eta \in i^b_c(\tau)]\!] \wedge [\![i^b_c(\eta) \in \tau]\!]
\end{aligned}$$

where the first line holds by 2.6.

Therefore, to show (1), we just need to show that for any $\eta \in dom(\tau)$, any $b \in C_\tau$, (second line holds by 2.7)

$$\begin{aligned}\llbracket \eta = i^b_c(\eta) \rrbracket &\geqslant \llbracket \eta \in i^b_c(\tau) \rrbracket \wedge \llbracket i^b_c(\eta) \in \tau \rrbracket \\ &= \bigvee_{\mu \in dom(\tau)} \tau(\mu) \wedge \llbracket \eta = i^b_c(\mu) \rrbracket \wedge \llbracket \mu = i^b_c(\eta) \rrbracket\end{aligned}$$

But this holds because for any $\mu \in dom(\tau)$,

$$\begin{aligned}\tau(\mu) \wedge \llbracket \eta = i^b_c(\mu) \rrbracket \wedge \llbracket \mu = i^b_c(\eta) \rrbracket &\leqslant \llbracket \eta = i^b_c(\mu) \rrbracket \wedge \llbracket \mu = i^b_c(\eta) \rrbracket \\ \text{(By 2.5)} \qquad &\leqslant \llbracket \eta = \mu \rrbracket \wedge \llbracket \mu = i^b_c(\eta) \rrbracket \\ &\leqslant \llbracket \eta = i^b_c(\eta) \rrbracket\end{aligned}$$

□

Theorem 2.9. $U^{\mathbb{B}} \models$ Replacement.

Proof. We may assume Collection does not hold in U, otherwise $U^{\mathbb{B}} \models$ Replacement as $U^{\mathbb{B}} \models$ Collection by Theorem 2.4. So we may assume there is a proper class of urelements in U.

It suffices to show that for every $\pi \in U^{\mathbb{B}}$, there is a $\rho \in U^{\mathbb{B}}$ such that for every $\sigma \in dom(\pi)$,

$$\llbracket \exists! y \varphi(\sigma, y) \rrbracket \leqslant \llbracket \exists y \in \rho\ \varphi(\sigma, y) \rrbracket \tag{2}$$

Fix a π and let $A = ker(\mathbb{B}) \cup ker(\pi)$.

Claim 2.9.1. For every $\sigma \in dom(\pi)$ and $\tau \in U^{\mathbb{B}}$, there is a $\tau^* \in U^{\mathbb{B}}$ such that $ker(\tau^*) \subseteq A$ and $\llbracket \varphi(\sigma, \tau) \wedge \forall z(\varphi(\sigma, z) \to z = \tau) \rrbracket \leqslant \llbracket \varphi(\sigma, \tau^*) \rrbracket$.

Proof of the Claim. Let $p = \llbracket \varphi(\sigma, \tau) \wedge \forall z(\varphi(\sigma, z) \to z = \tau) \rrbracket$ and c be a urelement such that $c \notin ker(\tau) \cup A$, which exists by our assumption. Then $\llbracket \varphi(\sigma, \tau) \rrbracket = \llbracket \varphi(\sigma, i^b_c(\tau)) \rrbracket$ for every $b \in ker(\tau) \setminus A$. Moreover, for each $b \in ker(\tau) \setminus A$,

$$\begin{aligned}p &\leqslant \llbracket \varphi(\sigma, \tau) \rrbracket \wedge (\llbracket \varphi(\sigma, i^b_c(\tau)) \rrbracket \Rightarrow \llbracket \tau = i^b_c(\tau) \rrbracket) \\ &\leqslant \llbracket \varphi(\sigma, \tau) \rrbracket \wedge \llbracket \tau = i^b_c(\tau) \rrbracket \\ &\leqslant \llbracket \tau = i^b_c(\tau) \rrbracket\end{aligned}$$

It follows that

$$\begin{aligned}p &\leqslant \llbracket \varphi(\sigma, \tau) \rrbracket \wedge \bigwedge_{b \in ker(\tau) \setminus A} \llbracket \tau = i^b_c(\tau) \rrbracket \\ \text{(by 2.8)} \qquad &\leqslant \llbracket \varphi(\sigma, \tau) \rrbracket \wedge \llbracket \tau = \overset{A}{\tau} \rrbracket \\ &\leqslant \llbracket \varphi(\sigma, \overset{A}{\tau}) \rrbracket\end{aligned}$$

As the kernel of $\overset{A}{\tau}$ is contained in A, this proves the claim. ■

Now for every $\sigma \in dom(\pi)$ and $p \in \mathbb{B}$ such that there is some $\tau \in U^{\mathbb{B}}$ with $p = \llbracket \varphi(\sigma, \tau) \wedge \forall z(\varphi(\sigma, z) \to z = \tau) \rrbracket$, let $\alpha_{\sigma,p}$ be the least α such that $\exists \tau^* \in V_\alpha(A)$ with $p \leqslant \llbracket \varphi(\sigma, \tau^*) \rrbracket$. Such α exists by the claim. Let $\gamma = \bigcup_{\langle \sigma, p \rangle \in dom(\pi) \times \mathbb{B}} \alpha_{\sigma,p}$ and $\rho = (V_\gamma(A) \cap U^{\mathbb{B}}) \times \{1\}$. It is easy to check that for every $\sigma \in dom(\pi)$, $\llbracket \exists! y \varphi(\sigma, y) \rrbracket \leqslant \llbracket \exists y \in \rho\ \varphi(\sigma, y) \rrbracket$, which completes the proof. □

Theorem 2.10 (The Fundamental Theorem of Boolean-Valued Model of $\mathrm{ZFCU_R}$)**.** Let U be a model of $\mathrm{ZFCU_R}$ and $\mathbb{B}$ be a complete Boolean-algebra in U.

(1) $U^{\mathbb{B}} \models \mathrm{ZFCU_R}$.
(2) $U^{\mathbb{B}} \models$ ZFCU if $U \models$ Collection.
(3) $U^{\mathbb{B}} \models$ Plenitude if $U \models$ Plenitude.
(4) $U^{\mathbb{B}} \models \neg$Plenitude if $U \not\models$ Plenitude.
(5) $U^{\mathbb{B}} \models$ Tail if $U \models$ Tail.

Proof. (1) follows from Theorem 2.2 and 2.9. (2) is Lemma 2.4.

For (3), suppose that Plenitude holds in U. It suffices to show tht for any $\tau \in U^{\mathbb{B}}$, $[\![Ord(\tau)]\!] = \bigvee_{\alpha \in ORD} [\![\tau = \check{\alpha}]\!] \leqslant [\![\exists x \subseteq \mathscr{A}(x \sim \tau)]\!] = \bigvee_{\sigma \in U^{\mathbb{B}}} [\![\sigma \subseteq \mathscr{A} \wedge \sigma \sim \tau]\!]$. $[\![Ord(\tau)]\!] = \bigvee_{\alpha \in ORD} [\![\tau = \check{\alpha}]\!]$. For any ordinal α, there is some set of urelements A such that $A \sim \alpha$ and so $[\![\check{A} \sim \check{\alpha}]\!] = 1$. Therefore, $[\![\tau = \check{\alpha}]\!] = [\![\tau = \check{\alpha}]\!] \wedge [\![\check{\alpha} \sim \check{A}]\!] \wedge [\![\check{A} \subseteq \mathscr{A}]\!] \leqslant [\![\check{A} \subseteq \mathscr{A} \wedge \check{A} \sim \tau]\!]$. This shows that, $\bigvee_{\alpha \in ORD} [\![\tau = \check{\alpha}]\!] \leqslant \bigvee_{\sigma \in U^{\mathbb{B}}} [\![\sigma \subseteq \mathscr{A} \wedge \sigma \sim \tau]\!]$.

For (4), suppose *for reductio* that $U \not\models$ Plenitude but $U^{\mathbb{B}} \not\models \neg$ Plenitude. Then in U, there is some λ such that there is no set of urelements of size λ. Since $[\![\exists \tau(\tau \text{ is a cardinal } \wedge \check{\lambda} < \tau)]\!] = 1$, $[\![\exists \tau(\tau \text{ is a cardinal } \wedge \check{\lambda} < \tau \wedge \exists \sigma \subseteq \mathscr{A}(\tau \preceq \sigma))]\!] \neq 0$. Hence for some $\tau, \sigma \in U^{\mathbb{B}}$, $[\![\tau \text{ is a cardinal } \wedge \check{\lambda} \in \tau \wedge \tau \preceq \sigma \wedge \sigma \subseteq \mathscr{A}]\!] \neq 0$. By Lemma 2.3, $[\![\sigma \subseteq \mathscr{A}]\!] = [\![\sigma \subseteq \check{A}_\sigma]\!]$, where $A_\sigma = \{a \in \mathscr{A} \mid a \in dom(\sigma)\}$. So $[\![\tau \text{ is a cardinal } \wedge \check{\lambda} \in \tau \wedge \tau \preceq \check{A}_\sigma]\!] \neq 0$. But A_σ is equinumerous with some κ less than λ, so $[\![\check{A}_\sigma \sim \check{\kappa} \wedge \check{\kappa} < \check{\lambda}]\!] = 1$. Therefore, $[\![\tau \text{ is a cardinal } \wedge \check{\lambda} < \tau \wedge \check{\kappa} < \check{\lambda} \wedge \tau \preceq \check{\kappa}]\!] \neq 0$, which is a contradiction.

For (5), first observe that over $\mathrm{ZFCU_R}$, if Plenitude fails but Collection holds, then every set of urelements will have a tail cardinal. For given a set A of urelements, we can fix the least cardinal κ such that there is no set of urelements of size κ that is disjoint from A; by Collection, there is a set y such that for every $\lambda < \kappa$, there is a $B \in y$ of size λ that is disjoint from A. Then consider the union of all the sets in y that are disjoint from A—its cardinality will be the tail cardinal of A. Now suppose that $U \models$ Tail, then clearly, $U \not\models$ Plenitude, and $U \models$ Collection by Theorem 1.1. It follows from (2) and (4) that $U^{\mathbb{B}} \models (\neg$ Plenitude $\wedge$ Collection$)$ and hence $U^{\mathbb{B}} \models$ Tail. □

2.3. **DC_κ-scheme in $U^{\mathbb{B}}$.** Every sequence of $\mathbb{B}$-names in U has a canonical $\mathbb{B}$-name in $U^{\mathbb{B}}$. Since for any $\tau, \sigma \in U^{\mathbb{B}}$, we can define

$$\{\tau\}^{\mathbb{B}} = \{\langle \tau, 1 \rangle\}$$
$$\{\tau, \sigma\}^{\mathbb{B}} = \{\langle \tau, 1 \rangle, \langle \sigma, 1 \rangle\}$$
$$\langle \tau, \sigma \rangle^{\mathbb{B}} = \{\{\tau\}^{\mathbb{B}}, \{\tau, \sigma\}^{\mathbb{B}}\}$$

Then if $f : \alpha \to U^{\mathbb{B}}$ is an α-sequence in U, we can define $\dot{f} \in U^{\mathbb{B}}$ as $\dot{f} = \{\langle \check{\beta}, f(\beta) \rangle^{\mathbb{B}} \mid \beta \in \alpha\} \times \{1\}$. It is easy to check the following.

Proposition 2.11. Let $f : \alpha \to U^{\mathbb{B}}$ be a function in U. $U^{\mathbb{B}} \models \dot{f}$ is a function on $\check{\alpha}$; for any $\beta < \alpha$, $U^{\mathbb{B}} \models \dot{f}(\check{\beta}) = f(\beta)$ and $U^{\mathbb{B}} \models (\dot{f \restriction \beta}) = \dot{f} \restriction \check{\beta}$. □

A partial order $(\mathbb{P}, \leqslant)$ is κ-closed if every descending chain in $\mathbb{P}$ of length $\lambda < \kappa$ in $\mathbb{P}$ has a lower bound. A complete Boolean algebra $\mathbb{B}$ is κ-closed if $(\mathbb{B}^+, \leqslant_{\mathbb{B}})$ has a dense subset that is κ-closed. It is a classic result that κ-closed $\mathbb{B}$ will preserve cardinals below κ: in particular, if $\mathbb{B}$ is κ-closed and $\omega_\alpha \leqslant \kappa$, then $U^{\mathbb{B}} \models \check{\omega_\alpha} = \omega_\alpha$. The next theorem shows that κ^+-closed $\mathbb{B}$ will preserve DC_κ-scheme over $\mathrm{ZFCU_R}$.

Theorem 2.12. Let κ be an infinite cardinal. If $U \models \mathrm{DC}_\kappa$-scheme and $\mathbb{B}$ is κ^+-closed in U, then $U^{\mathbb{B}} \models \mathrm{DC}_\kappa$-scheme.

Proof. Let $\kappa = \omega_\gamma$. Let $[\![\forall x \exists y \varphi(x,y)]\!] = p$. We need to show that

$$p \leqslant [\![\exists f(f \text{ is a function on } \omega_\gamma \wedge \forall \alpha < \omega_\gamma \varphi(f \restriction \alpha, f(\alpha)))]\!].$$

We have for any $\tau \in U^{\mathbb{B}}$, $p \leqslant \bigvee_{\sigma \in U^{\mathbb{B}}} [\![\varphi(\tau,\sigma)]\!]$. An infinite sequence $\langle\langle p_\alpha, \tau_\alpha\rangle : \alpha \in \kappa\rangle$ is said to be *a φ-chain below* p if $\langle p_\alpha : \alpha < \kappa\rangle$ is an infinite descending sequence below p and for every $\alpha < \kappa$, $p_\alpha \leqslant [\![\varphi(\dot{f_\alpha}, \tau_\alpha)]\!]$, where $f_\alpha = \langle \tau_\beta : \beta < \alpha\rangle$. A $q \in \mathbb{B}$ is said to bound $\langle\langle p_\alpha, \tau_\alpha\rangle : \alpha \in \kappa\rangle$ if q is a lower bound of $\langle p_\alpha : \alpha < \kappa\rangle$. Let $X = \{q \in \mathbb{B}^+ : q \text{ bounds some } \varphi\text{-chain below } p\}$.

Claim 2.12.1. X is dense below p.

Proof of the Claim. Let $0 \neq p' \leqslant p$, $D \subseteq \mathbb{B}^+$ be a dense subset that is κ^+-closed, and $D_{\leqslant p'} = \{q \in D \mid q \leqslant p'\}$. For every $\langle\langle p_\beta, \tau_\beta\rangle : \beta < \alpha\rangle \subseteq D_{\leqslant p'} \times U^{\mathbb{B}}$ such that $\langle p_\beta : \beta < \alpha\rangle$ is a descending chain, as D is κ-closed, there are $q \in \mathbb{B}$ and $\sigma \in U^{\mathbb{B}}$ such that $q \leqslant p' \wedge [\![\varphi(\dot{f_\alpha}, \sigma)]\!]$. By DC_κ-scheme in U, there exists a φ-chain below p', where all the first components of the pairs are in D. Since D is κ^+-closed, this chain has a bound in X. ■

Thus, $p \leqslant \bigvee X$. Let $q \in X$. Then q bounds some φ-chain $\langle\langle p_\alpha, \tau_\alpha\rangle : \alpha \in \kappa\rangle$ below p. Let $g : \kappa \to U^{\mathbb{B}}$ be the sequence $\langle \tau_\alpha : \alpha < \kappa\rangle$. $U^{\mathbb{B}} \models \dot{g}$ is a function on κ, and for any $\alpha < \kappa$, $U^{\mathbb{B}} \models \dot{g}(\check{\alpha}) = g(\alpha)$. And $q \leqslant p_\alpha \leqslant [\![\varphi((\dot{g \restriction \alpha}), g(\alpha))]\!]$ for all $\alpha < \kappa$, so $q \leqslant \bigwedge_{\alpha<\kappa} [\![\varphi(\dot{g} \restriction \check{\alpha}, \dot{g}(\check{\alpha}))]\!] = [\![\forall \alpha < \check{\kappa}(\varphi(\dot{g} \restriction \alpha, \dot{g}(\alpha)))]\!]$. $\mathbb{B}$ is κ-closed, so $U^{\mathbb{B}} \models \check{\kappa} = \omega_\gamma$. Thus, $q \leqslant [\![\exists f(f \text{ is a function on } \omega_\gamma \wedge \forall \alpha < \omega_\gamma \varphi(f \restriction \alpha, f(\alpha)))]\!]$. Therefore, $p \leqslant \bigvee X \leqslant [\![\exists f(f \text{ is a function on } \omega_\gamma \wedge \forall \alpha < \omega_\gamma \varphi(f \restriction \alpha, f(\alpha)))]\!]$. □

Let us note that the assumption of this theorem cannot be dropped. Given an infinite cardinal κ, consider the complete Boolean algebra $RO(\kappa^\omega)$ which consists of all the regular open sets of the product topology κ^ω, where κ is assigned the discrete topology. It is well-known that in $U^{RO(\kappa^\omega)}$, $\check{\kappa}$ is collapsed to ω. We include a proof of this fact for completeness.

Theorem 2.13 (folklore)**.** Let κ be an infinite cardinal and $\mathbb{B} = RO(\kappa^\omega)$. Then $U^{\mathbb{B}} \models \check{\kappa} \sim \check{\omega}$.

Proof. It suffices to show that $U^{\mathbb{B}} \models \check{\kappa} \leqslant \check{\omega}$. For each $n \in \omega$ and $\xi \in \kappa$, let $p_{n\xi} = \{g \in \kappa^\omega \mid g(n) = \xi\}$. Define $\tau \in U^{\mathbb{B}}$ as follows: $dom(\tau) = \{\langle \check{n}, \check{\xi}\rangle^{\mathbb{B}} \mid n \in \omega, \xi \in \kappa\}$, and for any $\langle \check{n}, \check{\xi}\rangle^{\mathbb{B}} \in dom(\tau)$, $\tau(\langle \check{n}, \check{\xi}\rangle^{\mathbb{B}}) = p_{n\xi}$.

Since for any $n \in \omega$, $\xi_1 \neq \xi_2 \in \kappa$, $p_{n\xi_1} \wedge p_{n\xi_2} = \{g \in \kappa^\omega \mid g(n) = \xi_1\} \cap \{g \in \kappa^\omega \mid g(n) = \xi_2\} = \emptyset$, $U^{\mathbb{B}} \models \tau$ is a partial function on $\check{\omega}$. Also, for any $\xi \in \kappa$, $\bigvee_{n<\omega} p_{n\xi} = (\overline{\{g \in \kappa^\omega \mid \text{for some } n < \omega, g(n) = \xi\}})^\circ = \kappa^\omega$. So for any $\xi \in \kappa$, $[\![\exists x \in \check{\omega}(\tau(x) = \check{\xi})]\!] = \bigvee_{n<\omega} p_{n\xi} = \kappa^\omega$. Therefore, $U^{\mathbb{B}} \models \tau$ is a surjection onto $\check{\kappa}$. □

Remark 2.14. $U^{\mathbb{B}}$ does not preserve DC_κ-scheme in general even if U satisfies ZFCU.

Proof. Consider a model U of $\mathrm{ZFCU_R}$ where every set of urelements has tail cardinal ω_1 (the existence of such model assuming the consistency of ZF is shown in [14]), and let $\mathbb{B} = RO(\omega_1^\omega)$. By Theorem 1.1 and 1.2, both DC_{ω_1}-scheme and Collection hold in U. We claim that there is an instance DC_{ω_1}-scheme that has value 0 in $U^{\mathbb{B}}$. For every $\tau \in U^{\mathbb{B}}$, $[\![\tau \subseteq \mathscr{A}]\!] = [\![\tau \subseteq \check{A}_\tau]\!]$ by Lemma 2.3, where A_τ is the set of urelements in $dom(\tau)$, and $[\![\check{A_\tau} \preceq \check{\omega_1}]\!]$ since every set of urelements in U has size at least ω_1. It then follows from the theorem above that $[\![\tau \subseteq \mathscr{A}]\!] \leq [\![\tau \preceq \omega]\!]$ for every $\tau \in U^{\mathbb{B}}$, i.e., $U^{\mathbb{B}} \models$ every set of urelements is countable. But over $\mathrm{ZFCU_R}$, if $\mathscr{A}$ is a proper class and every set of urelements is countable, then DC_{ω_1}-scheme fails. This is because in this case for every x there is a y with $ker(x) \subsetneq ker(y)$, but there cannot be a function f on ω_1 with $ker(f \restriction \alpha) \subsetneq ker(f(\alpha))$ for all $\alpha < \omega_1$ otherwise f would have an uncountable kernel. Therefore, the corresponding instance of DC_{ω_1}-scheme will have value 0 in $U^{\mathbb{B}}$. □

However, we do not know if $U^{\mathbb{B}}$ preserves DC_ω-scheme over $\mathrm{ZFCU_R}$ for every $\mathbb{B}$.

$RO(\kappa^\omega)$ can also be used to show that $U^{\mathbb{B}}$ does not preserve the failure of Collection. In other words, for certain models of $\mathrm{ZFCU_R}$, we can always recover Collection in some $U^{\mathbb{B}}$.

Theorem 2.15. Suppose that in U for every set of urelements, there is an infinite set of urelements disjoint from it. Then for some complete Boolean algebra $\mathbb{B} \in U$, $U^{\mathbb{B}} \models$ ZFCU.

Proof. We may assume that Plenitude fails in U otherwise by Theorem 1.1 and Theorem 2.10, every $U^{\mathbb{B}}$ satisfies ZFCU. Thus, there is a least cardinal κ in U such that there is no set of urelements of size κ. Let $\mathbb{B} = RO(\kappa^\omega)$. As in the previous remark, $U^{\mathbb{B}} \models$ every set of urelements is countable. Moreover, for every $\tau \in U^{\mathbb{B}}$, let $B \in U$ be an infinite set of urelements disjoint from A_τ, then since $[\![\check{A_\tau} \cap \check{B} = \emptyset]\!] = 1$, we have $[\![\tau \subseteq \mathscr{A}]\!] = [\![\tau \subseteq A_\tau]\!] \leqslant [\![\tau \cap \check{B} = \emptyset \wedge \check{B} \text{ is infinite}]\!]$. Therefore, $U^{\mathbb{B}} \models$ Tail because $U^{\mathbb{B}}$ thinks that every set of urelements has tail cardinal ω. By Theorem 1.1, it follows that $U^{\mathbb{B}} \models$ Collection. □

The assumption of this theorem cannot be dropped: if $\mathscr{A}$ is a proper class in U but every set of urelements is finite, then every $U^{\mathbb{B}}$ will inherit this property so Collection will fail in all $U^{\mathbb{B}}$.

Corollary 2.15.1. If $U \models \mathrm{DC}_\omega$-scheme, then $U^{\mathbb{B}} \models$ ZFCU for some $\mathbb{B} \in U$.

Proof. We may assume $\mathscr{A}$ is a proper class in U. Then by DC_ω-scheme, for every set A of urelements, there is an infinite sequence of sets of urelements $\langle A_n : n < \omega \rangle$ such that $A_n \subsetneq A_{n+1}$ and $A_n \cap A = \emptyset$. The kernel of such sequence will be an infinite set of urelements disjoint from A. So we can apply Theorem 2.15. □

3. Fullness of $U^{\mathbb{B}}$ and $\overline{U^{\mathbb{B}}}$

3.1. $U^{\mathbb{B}}$ **is almost never full.** An important property of Boolean-valued models is fullness.

Definition 3.1. Let $M^{\mathbb{B}}$ be a Boolean-valued model for the languague $\mathscr{L}$. $M^{\mathbb{B}}$ is full iff for any formula $\varphi(v, v_1, ..., v_n)$ in $\mathscr{L}$, any $x_1, ..., x_n \in M^{\mathbb{B}}$, there is some $x \in M^{\mathbb{B}}$ such that $[\![\exists v \varphi(v, x_1, ..., x_n)]\!] = [\![\varphi(x, x_1, ..., x_n)]\!]$.

In other words, a Boolean-valued model is full just in case there exists a "witness" in the model for every existential sentence. It is well known that if $V \models$ ZFC, then $V^{\mathbb{B}}$ is full for every $\mathbb{B}$. Fullness has many applications (e.g., see [11], [10], [2], and [13]). For example, it is well-known that given a full $M^{\mathbb{B}}$ and an ultrafilter on $\mathbb{B}$, we can prove the (generalized) Łoś theorem on $M^{\mathbb{B}}$, which, in the context of set theory, gives rise to a simple and direct way to obtain relative consistency results by forcing (see [7] for more on this). Without fullness, however, this cannot be done. Our $U^{\mathbb{B}}$ thus comes with a major drawback.

Remark 3.1. $U^{\mathbb{B}}$ is not full given that $\mathbb{B}$ is a proper extension of $\mathbb{2}$.

Proof. Consider, for example, the $\mathbb{B}$-name $\tau = \{\langle a_1, p\rangle, \langle a_2, \neg p\rangle\}$, where a_1, a_2 are two different urelements and p is an intermediate Boolean value. Let $\varphi(v)$ be the formula $\exists x(\mathscr{A}(x) \wedge x \in v)$. Then $[\![\varphi(\tau)]\!] = 1$. Suppose *for reductio* that for some $\sigma \in U^{\mathbb{B}}$, $[\![\mathscr{A}(\sigma)]\!] = 1$ and $[\![\sigma \in \tau]\!] = 1$. But by Definition 2.1, such σ must be identical to both a_1 and a_2. □

One root of the failure of fullness in $U^{\mathbb{B}}$ is that $U^{\mathbb{B}}$ lacks "mixtures" of $\mathbb{B}$-names. In $V^{\mathbb{B}}$, given an antichain $\{p_i \mid i \in I\} \subseteq \mathbb{B}$, and a sequence of names $\{x_i \mid i \in I\} \in V^{\mathbb{B}}$, we may construct their *mixture* $u \in V^{\mathbb{B}}$ as follows. We let $dom(u)$ be $\bigcup_{i\in I} dom(x_i)$ and for $y \in dom(u)$, $u(y) = \bigvee_{i \in I} p_i \wedge [\![y \in x_i]\!]$. Then we can prove the standard Mixing Lemma for $V^{\mathbb{B}}$, which suffices for fullness in $V^{\mathbb{B}}$.

Lemma 3.2 (The Mixing Lemma in $V^{\mathbb{B}}$)**.** Let V be a model of ZFC and $\mathbb{B} \in V$ be a complete Boolean algebra. For any antichain $\{p_i \mid i \in I\} \subseteq \mathbb{B}$ and any sequence of names $\{x_i \mid i \in I\} \in V^{\mathbb{B}}$, there is some $u \in V^{\mathbb{B}}$ (e.g., their mixture) such that for any $i \in I$, $p_i \leqslant [\![u = x_i]\!]$. □

The Mixing Lemma, however, does not hold in $U^{\mathbb{B}}$ because, as we have seen, Definition 2.1 does not even allow a mixture of two different urelements indexed by two incompatible intermediate values.

3.2. **A new construction:** $\overline{U^{\mathbb{B}}}$**.** Can there a different construction of Boolean-valued universe with urelements that satisfies the Mixing Lemma? Is there one that is full? We shall show that the answers to both questions are positive by providing a new construction.

Definition 3.2 ($\mathrm{ZFCU_R}$)**.** Let $\mathbb{B}$ be a complete Boolean algebra.

(1) $\tau : X \to \mathbb{B}$ is a $\overline{\mathbb{B}}$-name iff for any $x \in X$, x is either a urelement or a $\overline{\mathbb{B}}$-name, and for any urelement $a \in X$ and $x \in X$ such that $x \neq a$, $\tau(a) \wedge \tau(x) = 0$.

(2) Let τ be a $\overline{\mathbb{B}}$-name. $dom^{\mathscr{A}}(\tau) = \{a \in dom(\tau) \mid a \in \mathscr{A}\}$. $dom^{\mathbb{B}}(\tau) = \{\eta \in dom(\tau) \mid \eta \text{ is a } \overline{\mathbb{B}}\text{-name}\}$. If a is a urelement not in $dom^{\mathscr{A}}(\tau)$, then we let $\tau(a) = 0$.

(3) $\overline{U^{\mathbb{B}}} = \{\tau \in U : \tau \text{ is a } \overline{\mathbb{B}}\text{-name}\}$.

(4) $\mathcal{L}_{\overline{\mathbb{B}}}$ is the extended language that contains an additional binary predicate $\overset{\mathscr{A}}{=}$ and each $\mathbb{B}$-name as a constant symbol. $\mathcal{AL}_{\overline{\mathbb{B}}}$ is class of all atomic formulas in $\mathcal{L}_{\overline{\mathbb{B}}}$. The Boolean evaluation function $[\![\ \ ]\!]^{\overline{U^{\mathbb{B}}}} : \mathcal{AL}_{\overline{\mathbb{B}}} \to \mathbb{B}$ is defined as

follows.

$$\begin{aligned}
[\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} &= \bigvee_{a\in\mathscr{A}} \tau(a)\\
[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} &= \bigwedge_{a\in\mathscr{A}} (\tau(a) \Leftrightarrow \sigma(a))\\
[\![\tau \in \sigma]\!]^{\overline{U^{\mathbb{B}}}} &= \bigvee_{\mu\in dom^{\mathbb{B}}(\sigma)} [\![\tau = \mu]\!]^{\overline{U^{\mathbb{B}}}} \wedge \sigma(\mu)\\
[\![\tau \subseteq \sigma]\!]^{\overline{U^{\mathbb{B}}}} &= \bigwedge_{\eta\in dom^{\mathbb{B}}(\tau)} \tau(\eta) \Rightarrow [\![\eta \in \sigma]\!]^{\overline{U^{\mathbb{B}}}}\\
[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} &= [\![\tau \subseteq \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}}
\end{aligned}$$

$[\![\ ]\!]^{\overline{U^{\mathbb{B}}}}$ can be extended to all formulas in $\mathcal{L}_{\overline{\mathbb{B}}}$ in the standard way as before, and we shall let $\overline{U^{\mathbb{B}}}$ denote the structure $\left\langle \overline{U^{\mathbb{B}}}, [\![\ ]\!]^{\overline{U^{\mathbb{B}}}} \right\rangle$.

Let us note some differences between $U^{\mathbb{B}}$ and $\overline{U^{\mathbb{B}}}$. No urelement is a $\overline{\mathbb{B}}$-name since every $\overline{\mathbb{B}}$-name is a set, and each urelement a will be represented by $\{\langle a, 1\rangle\}$ in $\overline{U^{\mathbb{B}}}$ instead of itself. For any $\tau \in \overline{U^{\mathbb{B}}}$ and urelement a, $\tau(a)$ will be the $\mathbb{B}$-value of *a being identical* to τ, rather than the value of a's membership to τ. Accordingly, $[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}}$ is the degree to which τ and σ are identical when they are taken as urelements.

Proposition 3.3. For any τ, σ, π in $\overline{U^{\mathbb{B}}}$,

(i) $[\![\tau = \tau]\!]^{\overline{U^{\mathbb{B}}}} = 1$.
(ii) $\tau(\eta) \leqslant [\![\eta \in \tau]\!]^{\overline{U^{\mathbb{B}}}}$, for any $\eta \in dom^{\mathbb{B}}(\tau)$.
(iii) $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} = [\![\sigma = \tau]\!]^{\overline{U^{\mathbb{B}}}}$.
(iv) $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma = \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\tau = \pi]\!]^{\overline{U^{\mathbb{B}}}}$.
(v) $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\tau \in \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\sigma \in \pi]\!]^{\overline{U^{\mathbb{B}}}}$.
(vi) $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\pi \in \tau]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\pi \in \sigma]\!]^{\overline{U^{\mathbb{B}}}}$.
(vii) $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\mathscr{A}(\sigma)]\!]^{\overline{U^{\mathbb{B}}}}$

Proof. The proofs are all similar to the arguments in Theorem 1.23 in [3], and the clause on $[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}}$ causes no difficulties. For (iv), we need to show that $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma = \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\tau \subseteq \pi]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\pi \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}}$ and $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma = \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\tau \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}}$. The former is proved by the same argument as in [3, p.31]. And the latter holds because

$$\begin{aligned}
[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}} &= \bigwedge_{a\in\mathscr{A}} \tau(a) \Leftrightarrow \sigma(a) \wedge \bigwedge_{a\in\mathscr{A}} \sigma(a) \Leftrightarrow \pi(a)\\
&\leqslant \bigwedge_{a\in\mathscr{A}} \tau(a) \Leftrightarrow \pi(a) = [\![\tau \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}}
\end{aligned}$$

For (vii), it suffices to show that $[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\mathscr{A}(\sigma)]\!]^{\overline{U^{\mathbb{B}}}}$, which holds because

$$\bigwedge_{a\in\mathscr{A}} \tau(a) \Leftrightarrow \sigma(a) \wedge \bigvee_{a\in\mathscr{A}} \tau(a) \leqslant \bigvee_{a\in\mathscr{A}} \sigma(a)$$

□

Therefore, $\overline{U^{\mathbb{B}}}$ is indeed a Boolean-valued model.

Proposition 3.4. For any formula $\varphi(x)$ and any τ in $\overline{U^{\mathbb{B}}}$,

$$[\![\exists x \in \tau(\varphi(x))]\!]^{\overline{U^{\mathbb{B}}}} = \bigvee_{\eta \in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \wedge [\![\varphi(\eta)]\!]^{\overline{U^{\mathbb{B}}}}).$$

Proof. The proof is exactly the same as in [3, p.27]. One just need to note that the urelements in $dom(\sigma)$, if any, will be ignored when computing $[\![\tau \in \sigma]\!]^{\overline{U^{\mathbb{B}}}}$. □

We now verify that $\overline{U^{\mathbb{B}}}$ has mixtures.

Definition 3.3. Let $\{\tau_i \mid i \in I\} \subseteq \overline{U^{\mathbb{B}}}$ and $\{p_i \mid i \in I\} \subseteq \mathbb{B}$ be an antichain. We define the *mixture* of $\{\tau_i \mid i \in I\}$ with respect to $\{p_i \mid i \in I\}$ to be the $\overline{\mathbb{B}}$-name τ such that

$$dom(\tau) = \bigcup_{i \in I} dom(\tau_i)$$

For any $x \in dom(\tau)$, we define J_x as $\{i \in I \mid x \in dom(\tau_i)\}$, and define

$$\tau(x) = \bigvee_{i \in J_x} p_i \wedge \tau_i(x)$$

Proposition 3.5. Let $\{\tau_i \mid i \in I\} \subseteq \overline{U^{\mathbb{B}}}$ and $\{p_i \mid i \in I\} \subseteq \mathbb{B}$ be an antichain. Their mixture τ is a $\overline{\mathbb{B}}$-name.

Proof. Let $a \in dom^{\mathscr{A}}(\tau)$, $x \neq a \in dom(\tau)$. Then $\tau(a) \wedge \tau(v) = \bigvee_{i \in J_a} p_i \wedge \tau_i(a) \wedge \bigvee_{j \in J_x} p_j \wedge \tau_j(x)$. We need to show that for any $i \in J_a, j \in J_x$,

$$p_i \wedge \tau_i(a) \wedge p_j \wedge \tau_j(x) = 0$$

If $i \neq j$, then $p_i \wedge p_j = 0$ as $\{p_i \mid i \in I\}$ is an antichain. If $i = j$, then $a, x \in dom(\tau_i)$, and hence $\tau_i(a) \wedge \tau_i(x) = 0$ as τ_i is a $\overline{\mathbb{B}}$-name. □

Theorem 3.6 (The Mixing Lemma for $\overline{U^{\mathbb{B}}}$)**.** Let $\{\tau_i \mid i \in I\} \subseteq \overline{U^{\mathbb{B}}}$ and $\{p_i \mid i \in I\} \subseteq \mathbb{B}$ be an antichain and τ be their mixture. Then, for any $i \in I$,

$$p_i \leqslant [\![\tau = \tau_i]\!]^{\overline{U^{\mathbb{B}}}}$$

Proof. Let $i \in I$. We first show that

$$p_i \leqslant [\![\tau \subseteq \tau_i]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau)} \tau(\eta) \Rightarrow [\![\eta \in \tau_i]\!]^{\overline{U^{\mathbb{B}}}} \tag{3}$$

Since $\tau(\eta) = \bigvee_{j \in J_\eta} p_i \wedge \tau_j(\eta)$, we just need to show that for any $j \in J_\eta$,

$$p_i \leqslant \neg p_j \vee \neg \tau_j(\eta) \vee [\![\eta \in \tau_i]\!]^{\overline{U^{\mathbb{B}}}}$$

If $i \neq j$, then $p_i \leqslant \neg p_j$ as $\{p_i \mid i \in I\}$ is an antichain. If $i = j$, then $\eta \in dom(\tau_i)$. Hence $\tau_i(\eta) \leqslant [\![\eta \in \tau_i]\!]^{\overline{U^{\mathbb{B}}}}$ by 3.3(ii). Hence $\neg \tau_j(\eta) \vee [\![\eta \in \tau_i]\!]^{\overline{U^{\mathbb{B}}}} = 1$.

We next show that

$$p_i \leqslant [\![\tau_i \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau_i)} \tau_i(\eta) \Rightarrow [\![\eta \in \tau]\!]^{\overline{U^{\mathbb{B}}}} \tag{4}$$

Let $\eta \in dom^{\mathbb{B}}(\tau_i)$. Then $\eta \in dom^{\mathbb{B}}(\tau)$. Hence

$$\begin{aligned} p_i &\leqslant \tau_i(\eta) \Rightarrow (p_i \wedge \tau_i(\eta)) \\ &= \tau_i(\eta) \Rightarrow \tau(\eta) \\ &\leqslant \tau_i(\eta) \Rightarrow [\![\eta \in \tau]\!]^{\overline{U^{\mathbb{B}}}} \end{aligned}$$

We finally show that

$$p_i \leqslant [\![\tau \overset{\mathscr{A}}{=} \tau_i]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in \mathscr{A}} \tau(a) \Leftrightarrow \tau_i(a) \tag{5}$$

Let $a \in dom^{\mathscr{A}}(\tau)$. Since $\tau(a) = \bigvee_{j \in J_a} p_j \wedge \tau_j(a)$, we just need to show that the following two both hold:

$$p_i \leqslant \bigwedge_{j \in J_a} \neg p_j \vee \neg \tau_j(a) \vee \tau_i(a)$$

$$p_i \leqslant \neg \tau_i(a) \vee \bigvee_{i \in J_a} p_j \wedge \tau_j(a)$$

For the first statement, if $i \neq j$, then $p_i \leqslant \neg p_j$; if $i = j$, then $\neg \tau_i(a) \vee \tau_i(a) = 1$. For the second statement, if $a \notin dom^{(}\tau_i)$, then $\neg \tau_i(a) = 1$. If $a \in dom^{(}\tau_i)$, then $i \in J_a$. Hence $RHS \geqslant \neg \tau_i(a) \vee (p_i \wedge \tau_i(a)) = \neg \tau_i(a) \vee p_i \geqslant p_i$.

Combining (3) and (4) and (5) gives us what we want. □

3.3. $U^{\mathbb{B}}$ **and** $\overline{U^{\mathbb{B}}}$**.** It turns out that $U^{\mathbb{B}}$ can be elementarily embedded into $\overline{U^{\mathbb{B}}}$. To show this, we will first create an isomorphic copy of $U^{\mathbb{B}}$ within $\overline{U^{\mathbb{B}}}$ consisting of the "sharp" $\overline{\mathbb{B}}$-names. We will then show that this isomorphic copy is actually an elementary submodel of $\overline{U^{\mathbb{B}}}$.

Definition 3.4. Let $\tau \in \overline{U^{\mathbb{B}}}$. τ is a *sharp* $\overline{\mathbb{B}}$-name iff $\tau = \{\langle a, 1\rangle\}$, for some $a \in \mathscr{A}$, or for any $x \in dom(\tau)$, x is a sharp $\overline{\mathbb{B}}$-name. $\overline{U^{\mathbb{B}}_S}$ is the submodel of $\overline{U^{\mathbb{B}}}$ whose domain is the class of all the sharp $\overline{\mathbb{B}}$-names.

The following is easy to check.

Proposition 3.7. Let τ, σ be sharp $\overline{\mathbb{B}}$-names. Then,

(i) $[\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} = \begin{cases} 1 & \text{if for some urelement } a, \tau = \{\langle a, 1\rangle\}. \\ 0 & \text{if otherwise.} \end{cases}$

(ii) $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} = \begin{cases} 1 & \text{if } \tau = \sigma, \text{ and } \overline{U^{\mathbb{B}}} \models \mathscr{A}(\tau) \wedge \mathscr{A}(\sigma). \\ 0 & \text{if } \tau \neq \sigma, \text{ and } \overline{U^{\mathbb{B}}} \models \mathscr{A}(\tau) \vee \mathscr{A}(\sigma). \\ [\![\tau \subseteq \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} & \text{if } \overline{U^{\mathbb{B}}} \models \neg\mathscr{A}(\tau) \wedge \neg\mathscr{A}(\sigma). \end{cases}$

(iii) $[\![\tau \in \sigma]\!]^{\overline{U^{\mathbb{B}}}} = \begin{cases} 0 & \text{if } \overline{U^{\mathbb{B}}} \models \mathscr{A}(\tau). \\ \bigvee_{\eta \in dom(\sigma)} [\![\tau = \eta]\!]^{\overline{U^{\mathbb{B}}}} \wedge \sigma(\eta) & \text{if } \overline{U^{\mathbb{B}}} \models \neg\mathscr{A}(\tau). \end{cases}$

□

Lemma 3.8. $\overline{U^{\mathbb{B}}_S}$ and $U^{\mathbb{B}}$ are isomorphic.

Proof. The isomorphism is witnessed by $f : \overline{U_S^{\mathbb{B}}} \to U^{\mathbb{B}}$ defined recursively as follows: (let τ be a sharp $\overline{\mathbb{B}}$-name)

$$f(\tau) = \begin{cases} a & \text{if for some urelement } a, \tau = \{\langle a, 1\rangle\}. \\ \{\langle f(\eta), \tau(\eta)\rangle \mid \eta \in dom(\tau)\} & \text{if otherwise.} \end{cases}$$

It follows from Definition 2.1 and Proposition 3.7 that f is an isomorphism. □

We now show that $\overline{U_S^{\mathbb{B}}}$ is an elementary submodel of $\overline{U^{\mathbb{B}}}$. We first need a few lemmas.

Lemma 3.9. Let $\tau \in \overline{U^{\mathbb{B}}}$. Then, for some $\{\sigma_i \mid i \in I\} \subseteq \overline{U_S^{\mathbb{B}}} \cap V(ker(\tau) \cup ker(\mathbb{B}))$, some maximal antichain $\{p_i \mid i \in I\} \subseteq \mathbb{B}$,

$$p_i \leqslant [\![\tau = \sigma_i]\!]^{\overline{U^{\mathbb{B}}}} \text{ for any } i \in I.$$

Proof. The proof is by induction on τ. For every $\tau \in U^{\mathbb{B}}$, let B_τ denote $ker(\tau) \cup ker(\mathbb{B})$. The inductive hypothesis is that for any $\eta \in dom^{\mathbb{B}}(\tau)$, for some $\{\mu_j^\eta \mid j \in J_\eta\} \subseteq \overline{U_S^{\mathbb{B}}} \cap V(B_\eta)$, some maximal antichain $\{q_j^\eta \mid j \in J_\eta\} \subseteq \mathbb{B}$,

$$q_j^\eta \leqslant [\![\eta = \mu_j^\eta]\!]^{\overline{U^{\mathbb{B}}}} \text{ for any } j \in J_\eta.$$

We define $\pi \in \overline{U_S^{\mathbb{B}}}$ as follows:

$$dom(\pi) = \{\mu_j^\eta \mid \eta \in dom^{\mathbb{B}}(\tau), j \in J_\eta\}$$

For any $\nu \in dom(\pi)$, let $X_\nu = \{\langle \eta, j\rangle \mid \nu = \mu_j^\eta, \eta \in dom^{\mathbb{B}}(\tau), j \in J_\eta\}$. Then,

$$\pi(\nu) = \bigvee_{\langle \eta, j\rangle \in X_\nu} \tau(\eta) \wedge q_j^\eta$$

It is easy to check that $\pi \in V(B_\tau)$ and π is a sharp $\overline{\mathbb{B}}$-name.

We now show that $[\![\tau \subseteq \pi]\!]^{\overline{U^{\mathbb{B}}}} = 1$. That is, for any $\eta \in dom^{\mathbb{B}}(\tau)$, $\tau(\eta) \leqslant [\![\eta \in \pi]\!]^{\overline{U^{\mathbb{B}}}}$. This is because

$$\begin{aligned} [\![\eta \in \pi]\!]^{\overline{U^{\mathbb{B}}}} &= \bigvee_{\nu \in dom(x)} [\![\eta = \nu]\!]^{\overline{U^{\mathbb{B}}}} \wedge x(\nu) \\ &\geqslant \bigvee_{j \in J_\eta} [\![\eta = \mu_j^\eta]\!]^{\overline{U^{\mathbb{B}}}} \wedge x(\mu_j^\eta) \\ &\geqslant \bigvee_{j \in J_\eta} [\![\eta = \mu_j^\eta]\!]^{\overline{U^{\mathbb{B}}}} \wedge \tau(\eta) \wedge q_j^\eta \\ &= \bigvee_{j \in J_\eta} q_j^\eta \wedge \tau(\eta) = \tau(\eta) \end{aligned}$$

We next show that $[\![\pi \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} = 1$. That is, for any $\nu \in dom(\pi)$, $\pi(\nu) \leqslant [\![\nu \in \tau]\!]^{\overline{U^{\mathbb{B}}}}$. This is because

$$\begin{aligned}
[\![\nu \in \tau]\!]^{\overline{U^{\mathbb{B}}}} &= \bigvee_{\eta \in dom^{\mathbb{B}}(\tau)} \tau(\eta) \wedge [\![\nu = \eta]\!]^{\overline{U^{\mathbb{B}}}} \\
&\geqslant \bigvee_{\langle \eta, j\rangle \in X_\nu} \tau(\eta) \wedge [\![\mu_j^\eta = \eta]\!]^{\overline{U^{\mathbb{B}}}} \\
&\geqslant \bigvee_{\langle \eta, j\rangle \in X_\nu} \tau(\eta) \wedge q_j^\eta = \pi(\nu)
\end{aligned}$$

Finally we observe that $[\![\tau \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in \mathscr{A}} \neg\tau(a)$, since π is sharp. Hence $[\![\tau = \pi]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in \mathscr{A}} \neg\tau(a)$. Also, it is easy to check that for any $a_i \in dom^{\mathscr{A}}(\tau)$, $[\![\tau = \{\langle a_i, 1\rangle\}]\!] = \tau(a_i)$. Hence the statement holds as $\{\tau(a_i) \mid a_i \in dom^{\mathscr{A}}(\tau)\} \cup \{\bigwedge_{a \in \mathscr{A}} \neg\tau(a)\}$ is a maximal antichain in $\mathbb{B}$, by the definition of $\mathbb{B}$-names. □

Lemma 3.10. Let $\tau \in U^{\mathbb{B}}$. Then,

$$\bigvee_{\sigma \in \overline{U_S^{\mathbb{B}}}} [\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} = 1$$

Proof. By the previous lemma, for some $\{\sigma_i \mid i \in I\} \subseteq \overline{U_S^{\mathbb{B}}}$, some maximal antichain $\{p_i \mid i \in I\} \subseteq \mathbb{B}$, $p_i \leqslant [\![\tau = \sigma_i]\!]^{\overline{U^{\mathbb{B}}}}$ for any $i \in I$. Hence,

$$\bigvee_{\sigma \in \overline{U_S^{\mathbb{B}}}} [\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \geqslant \bigvee_{i \in I} [\![\tau = \sigma_i]\!]^{\overline{U^{\mathbb{B}}}} \geqslant \bigvee_{i \in I} p_i = 1$$

□

Theorem 3.11. $\overline{U_S^{\mathbb{B}}}$ is an elemantary submodel of $U^{\mathbb{B}}$. That is, for any formula $\varphi(v_1, ..., v_n)$, any $\sigma_1, ..., \sigma_n \in \overline{U_S^{\mathbb{B}}}$, $[\![\varphi(\sigma_1, ..., \sigma_n)]\!]^{\overline{U_S^{\mathbb{B}}}} = [\![\varphi(\sigma_1, ..., \sigma_n)]\!]^{\overline{U^{\mathbb{B}}}}$. Therefore, $U^{\mathbb{B}}$ is elementarily embedded in $\overline{U^{\mathbb{B}}}$.

Proof. By induction on the complexity of φ. The atomic cases are already covered since $\overline{U_S^{\mathbb{B}}}$ is a submodel of $U^{\mathbb{B}}$. The cases for connectives are straightforward. Let $\varphi(v_1, ..., v_n) = \exists v \psi(v, v_1, ..., v_n)$.

$$[\![\exists v \psi(v, \sigma_1, ..., \sigma_n)]\!]^{\overline{U_S^{\mathbb{B}}}} = \bigvee_{\sigma \in \overline{U_S^{\mathbb{B}}}} [\![\psi(\sigma, \sigma_1, ..., \sigma_n)]\!]^{\overline{U_S^{\mathbb{B}}}}$$

On the other hand,

$$\begin{aligned}
[\![\exists v \psi(v, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^{\mathbb{B}}}} &= \bigvee_{\tau \in U^{\mathbb{B}}} [\![\psi(\tau, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^{\mathbb{B}}}} \\
\text{(By 3.10)} \qquad &= \bigvee_{\tau \in U^{\mathbb{B}}} \bigvee_{\sigma \in \overline{U_S^{\mathbb{B}}}} [\![\psi(\tau, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} \\
&= \bigvee_{\sigma \in \overline{U_S^{\mathbb{B}}}} [\![\psi(\sigma, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^{\mathbb{B}}}}
\end{aligned}$$

Hence the case for quantifiers holds by inductive hypothesis. □

As a result, all the results in Section 2 also hold for $\overline{U^{\mathbb{B}}}$.

3.4. **Fullness and Collection.** One might attempt to conclude that $\overline{U^{\mathbb{B}}}$ is full since it is a standard result (see [7], [12]) that Boolean-valued models satisfying the Mixing Lemma are full. However, $\overline{U^{\mathbb{B}}}$ is a definable proper class inside U, so whether $\overline{U^{\mathbb{B}}}$ is full will also depend on what axioms hold in U. In fact, we shall prove the following.

Theorem 3.12. Over $\mathrm{ZFCU_R}$, the following are equivalent.

(1) The Axiom of Collection.
(2) For every complete Boolean algebra $\mathbb{B}$, $\overline{U^{\mathbb{B}}}$ is full.

The argument for (1) $\to$ (2) is standard. We include it here for completeness. Suppose that Collection holds. Fix a complete Boolean-algebra $\mathbb{B}$ and consider $[\![\exists x\varphi(x,\mu)]\!]$[3] for some φ and $\mu \in \overline{U^{\mathbb{B}}}$. Let $q = [\![\exists x\varphi(x,\mu)]\!]$ and $S = \{p \in \mathbb{B} : \exists \sigma \in \overline{U^{\mathbb{B}}}(p \leq [\![\varphi(\sigma,\mu)]\!])\}$. By Choice, S contains a maximal antichain $\{p_i : i \in I\}$ which is below q. By Collection, there is a set x such that for every p_i, there is a $\sigma \in x$ with $p_i \leq [\![\varphi(\sigma,\mu)]\!]$. We can then well-order x and choose a $\tau_i \in \overline{U^{\mathbb{B}}}$ for each $i \in I$ such that $p_i \leq [\![\varphi(\tau_i,\mu)]\!]$. Let τ be the mixture of $\{\tau_i : i \in I\}$ with respect to $\{p_i : i \in I\}$ as in Definition 3.3. By the Mixing Lemma for $\overline{U^{\mathbb{B}}}$, $p_i \leq [\![\tau = \tau_i]\!]$ for every i. Therefore, $p_i \leq [\![\varphi(\tau,\mu)]\!]$ for every i and since $\bigvee_{i\in I} p_i = q$, it follows that $[\![\exists x\varphi(x,\mu)]\!] = [\![\varphi(\tau,\mu)]\!]$. Hence, $\overline{U^{\mathbb{B}}}$ is full.

Now we show that the fullness of every $\overline{U^{\mathbb{B}}}$ implies Collection. To start, there is also a canonical way of representing U in $\overline{U^{\mathbb{B}}}$: for any urelement a, we let $\check{a} = \{\langle a, 1\rangle\}$; for any set x, $\check{x} = \{\langle \check{y}, 1\rangle \mid y \in x\}$. And the map $x \mapsto \check{x}$ preserves Δ_0 assertions as before.[4] The next lemma says that if $\mathbb{B}$ is atomic, then for every τ, $\overline{U^{\mathbb{B}}}$ thinks that τ is some $\check{x}$ for some x whose kernel is contained in $ker(\tau) \cup ker(\mathbb{B})$.

Lemma 3.13. Let $\mathbb{B}$ be an atomic complete Boolean algebra and $\tau \in \overline{U^{\mathbb{B}}}$. Then $\bigvee_{x \in V(ker(\tau)\cup ker(\mathbb{B}))} [\![\tau = \check{x}]\!] = 1$.

Proof. Since $\mathbb{B}$ is an atomic complete Boolean algebra, it is isomorphic to some powerset algebra $\mathcal{P}(I)$ ordered by $\subseteq$. For every $\eta \in U^{\mathbb{B}}$, let B_η denote $ker(\eta) \cup ker(\mathbb{B})$. We prove it by induction on τ with the inductive hypothesis that for any $\eta \in dom^{\mathbb{B}}(\tau)$, $\bigvee_{x\in V(B_\eta)} [\![\eta = \check{x}]\!] = 1$. So for any $i \in I$, there is a unique $v^i_\eta \in V(B_\eta)$ such that $i \in [\![\eta = \check{v}^i_\eta]\!]$.

For any $i \in I$, we define x_i as follows.

$$\begin{cases} x_i = a & \text{if } a \in dom^{\mathscr{A}}(\tau) \text{ and } i \in \tau(a) \\ x_i = \{v^i_\eta \mid i \in \tau(\eta)\} & \text{if otherwise} \end{cases}$$

x_i is well-defined by the definition of $\mathbb{B}$-names. Note that $x_i \in V(B_\tau)$ since $B_\tau = \bigcup_{\eta \in dom^{\mathbb{B}}(\tau)} B_\eta \cup dom^{\mathscr{A}}(\tau)$.

We now show that for any $i \in I$,

$$i \in [\![\tau = \check{x}_i]\!] = [\![\tau \subseteq \check{x}_i]\!] \wedge [\![\check{x}_i \subseteq \tau]\!] \wedge [\![\tau \overset{\mathscr{A}}{=} \check{x}_i]\!]$$

We first show that

$$i \in [\![\tau \subseteq \check{x}_i]\!] = \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau)} \tau(\eta) \Rightarrow [\![\eta \in \check{x}_i]\!]$$

[3]All the superscripts on $[\![]\!]$ will be omitted from now on since this will no longer cause confusion.

[4]We have used $\check{x}$ to denote two different things but this should cause no confusion as we will only work in $\overline{U^{\mathbb{B}}}$ from this point.

Let $\eta \in dom^{\mathbb{B}}(\tau)$. If $i \notin \tau(\eta)$, then $i \in \neg\tau(\eta) \subseteq \tau(\eta) \Rightarrow [\![\eta \in \check{x}_i]\!]$. If $i \in \tau(\eta)$, then $v^i_\eta \in x_i$, and hence $i \in [\![\eta = \check{v^i_\eta}]\!] \subseteq [\![\eta \in \check{x}_i]\!] \subseteq \tau(\eta) \Rightarrow [\![\eta \in \check{x}_i]\!]$.

We next show that

$$i \in [\![\check{x}_i \subseteq \tau]\!] = \bigwedge_{v^i_\eta \in x_i} [\![\check{v^i_\eta} \in \tau]\!]$$

Let $v^i_\eta \in x_i$. Then $i \in \tau(\eta)$. Hence $i \in \tau(\eta) \wedge [\![\eta = \check{v^i_\eta}]\!] \subseteq [\![\check{v^i_\eta} \in \tau]\!]$.

We finally show that

$$i \in [\![\tau \overset{\mathscr{A}}{=} \check{x}_i]\!] = \bigwedge_{a \in \mathscr{A}} \tau(a) \Leftrightarrow \check{x}_i(a)$$

Let $a \in \mathscr{A}$. $i \in \tau(a)$ iff $x_i = a$ iff $i \in \check{x}_i(a)$. Hence $i \in \tau(a) \Leftrightarrow \check{x}_i(a)$.

Since for any $i \in I$, $i \in [\![\tau = \check{x}_i]\!]$,

$$\bigvee_{i \in I} [\![\tau = \check{x}_i]\!] = 1$$

Hence the lemma is proved as $x_i \in V(B_\tau)$ for any $i \in I$. □

Lemma 3.14. Let $\mathbb{B}$ be an atomic complete Boolean algebra. Let $\varphi(v_1, ..., v_n)$ be a formula and $x_1, ..., x_n \in U$. Let $[\![\varphi(x_1, ..., x_n)]\!]^{\mathbb{2}} = 1$ iff $U \models \varphi(x_1, ..., x_n)$ and $[\![\varphi(x_1, ..., x_n)]\!]^{\mathbb{2}} = 0$ iff $U \models \neg\varphi(x_1, ..., x_n)$. Then,

$$[\![\varphi(x_1, ..., x_n)]\!]^{\mathbb{2}} = [\![\varphi(\check{x_1}, ..., \check{x_n})]\!]^{\overline{U^{\mathbb{B}}}}$$

Proof. By induction on the complexity of φ. The atomic cases are already covered. The cases for connectives are straightforward. Let $\varphi(v_1, ..., v_n) = \exists v \psi(v, v_1, ..., v_n)$. Then,

$$[\![\exists v \psi(v, x_1, ..., x_n)]\!]^{\mathbb{2}} = \bigvee_{x \in U} [\![\psi(x, x_1, ..., x_n)]\!]^{\mathbb{2}}$$

On the other hand,

$$\begin{aligned}
[\![\exists v \psi(v, x_1, ..., x_n)]\!]^{\overline{U^{\mathbb{B}}}} &= \bigvee_{\tau \in \overline{U^{\mathbb{B}}}} [\![\psi(\tau, \check{x_1}, ..., \check{x_n})]\!]^{\overline{U^{\mathbb{B}}}} \\
\text{(By Lemma 3.13)} \quad &= \bigvee_{\tau \in \overline{U^{\mathbb{B}}}} [\![\psi(\tau, \check{x_1}, ..., \check{x_n})]\!]^{\overline{U^{\mathbb{B}}}} \wedge \bigvee_{x \in U} [\![\tau = \check{x}]\!]^{\overline{U^{\mathbb{B}}}} \\
&= \bigvee_{x \in U} [\![\psi(\check{x}, \check{x_1}, ..., \check{x_n})]\!]^{\overline{U^{\mathbb{B}}}} \\
\text{(By induction hypothesis)} \quad &= \bigvee_{x \in U} [\![\psi(\check{x}, \check{x_1}, ..., \check{x_n})]\!]^{\mathbb{2}} \\
&= [\![\exists v \psi(v, x_1, ..., x_n)]\!]^{\mathbb{2}}
\end{aligned}$$

□

Proof of Theorem 3.12. (2) → (1). Assume that for every $\mathbb{B}$, $\overline{U^{\mathbb{B}}}$ if full and suppose that in U, $\forall x \in u \exists y \varphi(x, y)$ for some u. We wish to find some set A of urelements such that $\forall x \in u \exists y \in V(A) \varphi(x, y)$. This will suffice for Collection because since u is a set, we can find a large enough α such that for all $x \in u$, there is some $y \in V_\alpha(A)$ with $\varphi(x, y)$.

Let $\mathbb{B} = \mathcal{P}(u)$. In $\overline{U^{\mathbb{B}}}$, the $\mathbb{B}$-value of a formula is thus a subset of u. By Lemma 3.14, it follows that $\overline{U^{\mathbb{B}}} \models \forall x \in \check{u} \exists y \varphi(x, y)$. Let τ be the mixture of $\{\check{x} \mid x \in u\}$ with respect to the antichain $\{\{x\} \mid x \in u\}$. By the Mixing Lemma, $x \in [\![\tau = \check{x}]\!]$ for every $x \in u$. So $[\![\tau \in \check{u}]\!] = \bigvee_{x \in u} [\![\tau = \check{x}]\!] = 1$ and hence $[\![\exists y \varphi(\tau, y)]\!] = 1$. Since $\overline{U^{\mathbb{B}}}$ is full, there is some $\sigma \in \overline{U^{\mathbb{B}}}$ with $[\![\varphi(\tau, \sigma)]\!] = 1$. Let $A = ker(\mathbb{B}) \cup ker(\sigma)$. By Lemma 3.13,$\bigvee_{y \in V(A)} [\![\sigma = \check{y}]\!] = 1$.

If $x \in u$, then for some $y \in V(A)$, $x \in [\![\sigma = \check{y}]\!]$. So $x \in ([\![\tau = \check{x}]\!] \wedge [\![\varphi(\tau, \sigma)]\!] \wedge [\![\sigma = \check{y}]\!])$, thus $x \in [\![\varphi(\check{x}, \check{y})]\!]$. By Lemma 3.14, it follows that $[\![\varphi(\check{x}, \check{y})]\!]$ must be 1, so by Lemma 3.14 again, $U \models \varphi(x, y)$. Therefore, $U \models \forall x \in u \exists y \in V(A) \varphi(x, y)$, which completes the proof. □

In particular, over ZFCU_R, the failure of Collection implies the failure of fullness for some $\overline{U^{\mathbb{B}}}$, where $\mathbb{B}$ is atomic. $\overline{U^{\mathbb{B}}}$ can also fail to be full for some non-atomic complete Boolean algebra when Collection fails. We shall end by providing an example of this sort.

Let U be such that $\mathscr{A}$ is a proper class but every set of urelements is finite and let $\mathbb{B} = RO(2^\omega)$. Define $\rho \in \overline{U^{\mathbb{B}}}$ as follows. $dom(\rho) = \{\check{n} \mid n \in \omega\}$, and for any $n \in \omega$, $\rho(\check{n}) = \{f \in 2^\omega \mid f(n) = 1\}$. It is routine to compute that for every $n \in \omega$ and $f \in 2^\omega$, $[\![\check{n} \in \rho]\!] = \{f \in 2^\omega \mid f(n) = 1\}$ and $[\![\check{n} \notin \rho]\!] = \{f \in 2^\omega \mid f(n) = 0\}$. ρ is then a set of natural numbers in $\overline{U^{\mathbb{B}}}$. And $\overline{U^{\mathbb{B}}}$ thinks that for some set of urelements, its cardinality is not in ρ, i.e.,

Claim 3.14.1. $\overline{U^{\mathbb{B}}} \models \exists A \subseteq \mathscr{A} \exists n \in \check{\omega}(A \sim n \wedge n \notin \rho))$.

Proof. Since in U, for every $n < \omega$ there is a set of urelements of size n, $\overline{U^{\mathbb{B}}}$ will think the same. So we only need to show that $\overline{U^{\mathbb{B}}} \models \exists n \in \check{\omega}(n \notin \rho)$, i.e., $\bigvee_{n \in \omega} [\![\check{n} \notin \rho]\!] = 2^\omega$. Let $f \in 2^\omega$. But

$$\bigvee_{n \in \omega} [\![\check{n} \notin \rho]\!] = (\overline{\{f \in 2^\omega \mid \text{for some } n \in \omega, f(n) = 0\}})^\circ = 2^\omega.$$

□

Claim 3.14.2. $\overline{U^{\mathbb{B}}}$ is not full.

Proof. Suppose *for reductio* that $\overline{U^{\mathbb{B}}}$ is full. It follows that there is some $\tau \in \overline{U^{\mathbb{B}}}$ such that

$$\overline{U^{\mathbb{B}}} \models \tau \subseteq \mathscr{A} \wedge \exists n \in \check{\omega}(\tau \sim n \wedge n \notin \rho).$$

So $ker(\tau) = \{a_1, ..., a_m\}$ will have m urelements for some finite number m. Then $\overline{U^{\mathbb{B}}} \models \forall x \in \tau(y = \check{a_1} \vee ... \vee y = \check{a_m})$. This is because for any $\eta \in dom^{\mathbb{B}}(\tau)$, $\tau(\eta) \leqslant [\![\mathscr{A}(\eta)]\!] = \bigvee_{a \in \mathscr{A}} \eta(a) = \bigvee_{a \in \mathscr{A}} [\![\eta = \check{a}]\!]$ and for any urelement b not in $ker(\tau)$, $[\![\eta = \check{b}]\!] = \eta(b)$, which is 0. Thus, $\tau(\eta) \leqslant [\![\mathscr{A}(\eta)]\!] \leqslant \bigvee_{i \leqslant m} [\![\eta = \check{a_i}]\!]$ for every $\eta \in dom^{\mathbb{B}}(\tau)$, and this implies that $\overline{U^{\mathbb{B}}} \models \forall x \in \tau(y = \check{a_1} \vee ... \vee y = \check{a_m})$.

Therefore, $\overline{U^{\mathbb{B}}} \models \tau \preceq \check{m}$. Let $M = \{f \in 2^\omega \mid \text{for any } i \leqslant m, f(m) = 1\} \in RO(2^\omega)$ and D be an ultrafilter on $RO(2^\omega)$ with $M \in D$. For any $i \leqslant m$,

$$[\![\check{i} \in \rho]\!] = \{f \in 2^\omega \mid f(i) = 1\} \supseteq M \in D$$

$\overline{U^{\mathbb{B}}}$ is full so for some $\sigma \in \overline{U^{\mathbb{B}}}$, we have $[\![\tau \sim \sigma \wedge \sigma \leqslant \check{m} \wedge \sigma \notin \rho]\!] = 1$. So $[\![\sigma \leqslant \check{m}]\!] = \bigvee_{i \leqslant m} [\![\sigma = \check{i}]\!] = 1 \in D$, and thus for some $i \leqslant m$, $[\![\sigma = \check{i}]\!] \in D$. It follows that $[\![\sigma \in \rho]\!] \in D$, which contradicts $[\![\sigma \notin \rho]\!] = 1$. □

However, we do not know if over $\mathrm{ZFCU_R}$, Collection is also equivalent to the claim that for every non-atomic $\mathbb{B}$, $\overline{U^{\mathbb{B}}}$ is full.

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